AN INTEGRAL EQUATION ASSOCIATED WITH LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. Associated with each linear homogeneous differential equation

\[ y^{(n)} = \sum_{i=0}^{n-1} a_i(x)y^{(i)} \quad \text{of order } n \quad \text{on the real line}, \]

there is an equivalent integral equation

\[ f(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} \left[ \int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f(v)dv \right] du, \]

which is satisfied by each solution \( f(x) \) of the differential equation.

KEY WORDS AND PHRASES. Linear homogeneous differential equations, Integral equations, Initial value problems, Variation of parameters formula, Uniform convergence.

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1. INTRODUCTION.

Let \( n \) be a positive integer, \( I \) be an interval on the real line \( \mathbb{R} \) and \( C(I) \) be the class of all functions continuous on \( I \). Let

\[ y^{(n)} = a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y, \quad x \in I \quad (1.1) \]

be any \( n \)-th order (normalized) ordinary linear homogeneous differential equation, where \( a_i(x) \in C(I), \ i=0,1,2,\ldots,(n-1). \)

The purpose of this article is to derive an equivalent integral equation satisfied by the solutions of the linear homogeneous differential equation (1.1).

2. MAIN RESULTS.

THEOREM. Let \( f(x) \) be a solution of (1.1) defined on \( I \) and \( x_0 \in I \). Then \( f(x) \) is also a solution of the integral equation

\[ f(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} \left[ \int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f(v)dv \right] du, \quad (2.1) \]

where \( h(x) \) is the unique solution of the \((n-1)\)-th order linear homogeneous differential equation

\[ y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + \ldots + a_1(x)y, \quad x \in I \quad (2.2) \]
satisfying the initial conditions
\[ y(x_0) = f'(x_0), \quad y'(x_0) = f''(x_0), \ldots, \quad y^{(n-2)}(x_0) = f^{(n-1)}(x_0), \]
and \( G_{n-1}(x,u) \) is the well-known Green's Function associated with the homogeneous equation (2.2).

**PROOF OF THE THEOREM.** In order to deduce the integral equation (2.1), we will use the well-known Variation of Parameters formula

\[ y(x) = h(x) + \int_{x_0}^{x} G_n(x,u) a_0(u)du \tag{2.3} \]

solving uniquely the non-homogeneous initial value problem

\[ y^{(n-1)}(x) = a_{n-1}(x)y^{(n-2)}(x) + \ldots + a_1(x)y + a_0(x) \]
\[ y(x_0) = f'(x_0), \quad y'(x_0) = f''(x_0), \ldots, \quad y^{(n-2)}(x_0) = f^{(n-1)}(x_0) \]
for each \( a \in \mathbb{C}(I) \).

Consider the sequence of functions:

\[ f_1(x), f_2(x), \ldots, f_k(x), \ldots \]

defined on \( I \), where

\[ f_1(x) = f(x_0) + \int_{x_0}^{x} h(u)du \]
\[ f_2(x) = f_1(x) + \int_{x_0}^{x} \left[ \int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f_1(v)dv \right]du \]
\[ \ldots \]
\[ f_k(x) = f_1(x) + \int_{x_0}^{x} \left[ \int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f_{k-1}(v)dv \right]du \]
\[ \ldots \]

Clearly, for each \( k \), \( f_k(x_0) = f(x_0) \), \( f_k(x) \) is differentiable on \( I \) and for \( k \geq 2 \)

\[ f'_k(x) = h(x) + \int_{x_0}^{x} G_{n-1}(x,u)a_0(u)f_{k-1}(u)du. \tag{2.5} \]

Using (2.3), we conclude that, for each \( k \geq 2 \), \( f'_k(x) \) is the unique solution of the non-homogeneous initial value problem

\[ y^{(n-1)}(x) = a_{n-1}(x)y^{(n-2)}(x) + \ldots + a_1(x)y + a_0(x) f_{k-1}(x) \]
\[ y(x_0) = f'(x_0), \quad y'(x_0) = f''(x_0), \ldots, \quad y^{(n-2)}(x_0) = f^{(n-1)}(x_0) \tag{2.6} \]

Hence each \( f_k(x) \in \mathbb{C}^n(I) \). Both the sequences \( \{f_k(x)\} \) and \( \{f'_k(x)\} \) converge uniformly on every compact subset on the interval \( I \). To see this, let \( B \) be a compact subset of \( I \). Then there exists a closed and bounded interval \([a,b]\) such that \( B \subset [a,b] \subset I \) and
Let $M = \max |G_{n-1}(u,v)a_0(v)|$, $s = \max |f_1(v)|$ for each $u,v \in [a,b]$. One can now see very easily that for each $x \in [a,b]$,

$$
|f_{k+1}(x) - f_k(x)| \leq M^k s^k (b-a)^{2k/2k!}
$$

$$
|f_{k+1}(x) - f'_k(x)| \leq M^k s^k (b-a)^{k/k!}
$$

using recursively the bounds for $|f_{i+1}(x) - f_i(x)|$, $i = 1,2,3,...$. Since each of the series $\sum k^k s^k (b-a)^{2k/2k!}$, $\sum k^k s^k (b-a)^{k/k!}$ converges, we conclude by Weierstrass' M-test that each of the series of functions

$$
f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)), f'_1(x) + \sum_{k=1}^{\infty} (f'_{k+1}(x) - f'_k(x))
$$

converges uniformly on $[a,b]$ and hence on $B$. Therefore there is a function $g(x) \in C^1(I)$ such that

$$
\lim_{k \to \infty} f_k(x) = g(x)
$$

$$
\lim_{k \to \infty} f'_k(x) = g'(x)
$$

for all $x \in I$. In particular $g(x_0) = f(x_0)$.

Also from (2.5) we get by taking limit as $k \to \infty$

$$
g(x) = h(x) + \int_{x_0}^{x} G_{n-1}(x,u)a_0(u)g(u)du,
$$

(2.7)

Hence

$$
g(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} \int_{x_0}^{u} G_{n-1}(u,v)a_0(v)g(v)dvdu
$$

(2.8)

Again, relation (2.7) implies by (2.3) that $g(x)$ is the unique solution of the initial value problem

$$
y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + ... + a_1(x)y + a_0(x)g(x)
$$

$$
y^{(n-1)}(x_0) = f^{(n-1)}(x_0).
$$

Therefore $g(x) \in C^n(I)$ and

$$
g^{(n)}(x) = a_{n-1}(x)g^{n-1}(x) + ... + a_0(x)g(x),
$$

$\forall x \in I$.

In other words, $g(x)$ is the unique solution of the homogeneous initial value problem

$$
y^{(n)} = a_{n-1}(x)y^{(n-1)}(x) + ... + a_0(x)y, \quad x \in I.
$$

$$
y(x_0) = f(x_0), \quad y^{(n-1)}(x_0) = f^{(n-1)}(x_0).
$$

Hence $f(x) = g(x)$ for all $x \in I$. Therefore, by (2.8)

$$
f(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} \int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f(v)dvdu.
$$
This completes the proof.

REMARK. The above proof clearly shows how a solution of a linear homogeneous equation with prescribed initial values can be constructed out of a solution $h(x)$ and the Green's Function $G_{n-1}(x,u)$ of a lower order homogeneous linear equation. This is specially significant in case of second order homogeneous equations, as solutions $\{ce^{A(x)}\}$ and the Green's function $G_1(x,u) = e^{A(x)-A(u)},$

\[ [A(x) = \int_{x_0}^{x} a_1(u)du], \]

of first order homogeneous equation $y = a_1(x)y$ are readily available.

REFERENCES
