A DIGRAPH EQUATION FOR HOMOMORPHIC IMAGES

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ABSTRACT. The definitions of a homomorphism and a contraction of a graph are generalized to digraphs. Solutions are given to the graph equation $\hat{\phi}(D) = \theta_\phi(\overline{D})$.

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By a graph $G$ we mean a finite graph with no multiple edges or loops. If graphs $G$ and $H$ are isomorphic we write $G \cong H$. An elementary homomorphism of a graph $G$ is an identification of two non-adjacent vertices of $G$ and a homomorphism is a sequence of elementary homomorphisms. A homomorphism of $G$ onto $H$ preserves adjacency. Likewise, an elementary contraction of $G$ is the identification of two adjacent vertices of $G$ and a contraction is a sequence of elementary contractions\[1\]. Thus for every homomorphism $\phi$ of $G$ there is a related contraction $\theta_\phi$ of the complement of $G$, $\overline{G}$. This contraction is constructed as follows: $\phi$ is a sequence of elementary homomorphisms $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ so we let $\theta_\phi$ be sequence of elementary contractions $\theta_1, \theta_2, \ldots, \theta_n$ where $\theta_i$ identifies the same vertices in $\overline{G}$ that $\epsilon_i$ identifies in $G$.

Recently\[2\] the graph equation $\hat{\phi}(G) = \theta_\phi(\overline{G})$ was studied. In this paper, we generalize the definition of a homomorphism and its related contraction to digraphs and find general solutions to this graph equation. In doing so, we find an easier proof of the result given in\[2\].

A digraph $D$ consists of a finite vertex set $V(D)$ together with a set $E(D)$ of ordered pairs of distinct elements of $V(D)$, called arcs. Again, if $D_1$ is isomorphic to $D_2$ we write $D_1 \cong D_2$. By an elementary homomorphism of $D$ we mean an identification of two mutually non-adjacent vertices of $D$ (neither $uv$ nor $vu$ are in $E(D)$). Similarly, an elementary contraction is an identification of two mutually adjacent vertices of $D$ (both $uv$ and $vu$ are in $E(D)$). A homomorphism (contraction) of $D$ is again a sequence of elementary homomorphisms (contractions). The contraction $\theta_\phi$ of $\overline{D}$ related to the homomorphism $\phi$ of $D$ is defined as for undirected graphs.
We will use the following notation as need arises: \( I_b(u) \) is the set of vertices \( v \) of \( D \) such that \( vu \) is an arc of \( D \), \( O_b(u) \) is the set of vertices \( v \) of \( D \) such that \( uv \) is an arc of \( D \), and \( A(u) \) is the adjacency set of \( u \) in the graph \( G \).

**Theorem 1.** Let \( \varepsilon \) be an elementary homomorphism of \( D \) identifying vertices \( u_1 \) and \( u_2 \). Then \( \varepsilon(D) = \varepsilon(D) \) if and only if \( I_b(u_1) = I_b(u_2) \) and \( O_b(u_1) = O_b(u_2) \).

**Proof.** Let \( u = \varepsilon(u_1) = \varepsilon(u_2) \). First suppose that \( O_b(u_1) \neq O_b(u_2) \). Excluding \( u \) as a possible endpoint of an arc, we have \( vv' \) is an arc of \( \varepsilon(D) \) if and only if \( vv' \) is an arc of \( \varepsilon(D) \). Hence there is a one to one correspondence of those arcs in \( \varepsilon(D) \) without \( u \) as an endpoint and those of \( \varepsilon(D) \) without \( u \) as an endpoint. The vertex \( v \) of the arc uv must be in \( O_b(u_1) \cap O_b(u_2) \), \( O_b(u_1) \cup O_b(u_2) \), or \( O_b(u_1) \setminus O_b(u_2) \), the symmetric difference. In the first case, \( uv \) is not an arc of \( \varepsilon(D) \) or \( \varepsilon(D) \), and in the second case, \( uv \) is an arc of both. The latter case implies that \( uv \) is not an arc of \( \varepsilon(D) \) but is an arc of \( \varepsilon(D) \). Thus for every vertex in \( O_b(u_1) \setminus O_b(u_2) \), \( \varepsilon(D) \) has one more arc than \( \varepsilon(D) \). The same holds for vertices in \( I_b(u_1) \setminus I_b(u_2) \). Thus if \( O_b(u_1) \neq O_b(u_2) \) or \( I_b(u_1) \neq I_b(u_2) \), \( |E(\varepsilon(D))| > |E(\varepsilon(D))| \) and hence \( \varepsilon(D) \neq \varepsilon(D) \). Now let \( I_b(u_1) = I_b(u_2) \) and \( O_b(u_1) = O_b(u_2) \). We will use the identity map from \( V(\varepsilon(D)) \) onto \( V(\varepsilon(D)) \) and hence need only consider arcs to and from \( u \). If \( uv \) is in \( E(\varepsilon(D)) \) then \( u_1v \) and \( u_2v \) are arcs in \( D \). Thus \( u_1v \) and \( u_2v \) are not arc of \( D \) and subsequently \( uv \) is in \( E(\varepsilon(D)) \). By the same argument, if \( uv \) is an arc of \( \varepsilon(D) \), \( uv \) will be an arc of \( \varepsilon(D) \). This holds for arcs \( vu \), so \( \varepsilon(D) = \varepsilon(D) \).  

**Corollary 1:** \( \varphi(D) = \varphi(D) \) if and only if \( \phi \) is a sequence of elementary homomorphisms, each of which satisfies the conditions of Theorem 1.

A digraph \( D \) is pseudo-complete \( n \)-partite if there is a partition \( V_1, V_2, \ldots, V_n \) such that \( u, u' \) in \( V_i \) for some \( i \) implies \( u \) and \( u' \) are mutually non-adjacent, if \( u \) is an element of \( V_i \) and \( v \) is an element of \( V_j \), \( i \neq j \), then either \( uv \) or \( vu \) is an arc of \( D \), and finally if \( u \) and \( u' \) are in \( V_i \), \( v \) and \( v' \) are in \( V_j \), \( i \neq j \), and \( uv \) is an arc then \( uv' \), \( u'v \), and \( u'v' \) are also.

**Theorem 2.** \( \varphi(D) = \varphi(D) \) for all homomorphisms \( \phi \) of \( D \) if and only if \( D \) is pseudo-complete \( n \)-partite.

**Proof.** If \( D \) is pseudo-completely \( n \)-partite, every elementary homomorphism identifies two vertices \( u_1 \) and \( u_2 \) in the same partition set and thus \( I_b(u_1) = I_b(u_2) \) and \( O_b(u_1) = O_b(u_2) \). Hence \( \varepsilon(D) = \varepsilon(D) \) for every elementary homomorphism and thus for every homomorphism of \( D \). Conversely, partition \( V(D) \) according to the relation: \( u_1 \) and \( u_2 \) are in \( V_i \) if and only if \( I_b(u_1) = I_b(u_2) \) and \( O_b(u_1) = O_b(u_2) \). We need only show that if \( u_1 \) is in \( V_i \) and \( u_2 \) is in \( V_j \), \( i \neq j \), then either \( u_1u_2 \) or \( u_2u_1 \) is in \( E(D) \). Suppose \( u_1 \) and \( u_2 \) are mutually non-adjacent and let \( \varepsilon \) be the elementary homomorphism identifying them. Since \( \varepsilon(D) = \varepsilon(D) \), \( O_b(u_1) = O_b(u_2) \) and \( I_b(u_1) = I_b(u_2) \) by Theorem 1 and hence \( u_1 \) and \( u_2 \) are in the same partition set. Thus if \( u_1 \) is in \( V_i \) and \( u_2 \) is in \( V_j \), \( i \neq j \), there is an arc between them and \( D \) must be pseudo-complete \( n \)-partite.
If, for every vertex $u$ of $D$, $I_b(u) = O_b(u)$, $D$ is a symmetric digraph and can be represented by a graph $G$. This leads to the following corollaries to Theorems 1 and 2.

**COROLLARY 2.** An elementary homomorphism $\epsilon$ identifying vertices $u$ and $v$ of a graph $G$ satisfies $\epsilon(G) = \theta_{\epsilon}(\overline{G})$ if and only if $\lambda(u_1) = \lambda(u_2)$.

**COROLLARY 3.** A homomorphism $\phi$ of $G$ satisfies $\phi(G) = \theta_{\phi}(\overline{G})$ if and only if $\phi$ is a sequence of elementary homomorphisms, each satisfying Corollary 2.

**COROLLARY 4.** $\phi(G) = \theta_{\phi}(\overline{G})$ for every homomorphism $\phi$ of $G$ if and only if $G$ is complete $n$-partite.

A study of the equation $\phi(D) = \theta_{\phi}(\overline{D})$ would be interesting, yet is apparently difficult considering the work done in [2] for graphs. We conjecture that if $D = \overline{D}$ and $\phi(D) = \theta_{\phi}(\overline{D})$, $\phi$ nontrivial, then $D$ is a symmetric digraph.

**REFERENCES**

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