NOTES ON ALMOST-PERIODICITY
IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. A study is made of almost-periodic functions in topological vector spaces with applications to abstract differential equations.

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1. INTRODUCTION.

In our recent papers [1, 2], we extended the theory of almost-periodic functions from Banach spaces to topological vector spaces and gave a few results concerning its applications to abstract differential equations. The following results are the continuation of discussions begun there. Specifically Theorem 2 is a version of a theorem contained in [1, 2] (see Theorem 5.1 in [2]) which was originally inspired from a result due to A. I. Perov (cf. [3] Theorem 1.1).

Let us first recall some useful facts (see [1, 2] for more details). The reader can also find in [4] the elementary properties of linear topological spaces needed here.

DEFINITION 1. A continuous function $f: \mathbb{R} \to E$, where $E$ is a complete locally convex space and $\mathbb{R}$ is the set of real numbers, is called almost periodic (a.p.) if for each neighborhood (of the origin in $E$) $U$, there exists a real number $\tau = \tau(U) > 0$ such that every interval $[a, a + \tau]$ contains at least a point $t$ such that

$$f(t + \tau) - f(t) \in U$$

for every $t \in \mathbb{R}$.

$\tau$ is then called a $U$-translation number of the function $f$.

REMARK. $U = U(\epsilon; p_i; \Lambda \leq i \leq N)$

$$= \{x \in E; p_i(x) < \epsilon, 1 \leq i \leq N\}$$

where each $p_i \in \mathcal{Q}$, the set of semi-norms on $E$.

Finally we recall Bochner's criteria: If $E$ is a Frechet space, then a function $f: \mathbb{R} \to E$ is a.p. iff for every real sequence $(s_n)_{n=1}^{\infty}$ there exists a subsequence $(s_{n_i})_{n=1}^{\infty}$ such that $(f(t + s_{n_i}))_{n=1}^{\infty}$ converges uniformly in $t \in \mathbb{R}$.

DEFINITION 2. A Frechet space $E$ is called a perfect Frechet space if the following property is verified in $E$: every function $\phi: \mathbb{R} \to E$ such that
(i) \( \{ \phi(t); t \in \mathbb{R} \} \) is bounded in \( E \)
(ii) the derivative \( \phi'(t) \) is a.p. in \( E \), is necessarily a.p. in \( E \).

2. MAIN RESULTS.

Now let us state and prove:

THEOREM 1. If \( f(t) \) is a.p. in a complete locally convex space \( E \), then for every real sequence \( (s_n)_{n=1}^{\infty} \) there exists a subsequence \( (s'_{n})_{n=1}^{\infty} \) such that for every neighborhood (of the origin in \( E \)) \( U \),

\[
f(t + s'_{n}) - f(t + s'_{m}) \in U
\]

for all \( t \in \mathbb{R} \), \( m \) and \( n \).

PROOF. Let \( U = U(\epsilon; p_i, 1 \leq i \leq n) \) be a neighborhood and \( V = V(\delta; p_i, 1 \leq i \leq n) \) a symmetric neighborhood such that \( V + V + V + V \subseteq U \). By the definition of almost-periodicity, there exists \( \varepsilon = \varepsilon(V) \) (therefore \( \varepsilon \) depends on \( U \)) such that in every real interval of length \( \varepsilon \), there exists \( \tau \) such that

\[
f(t + \tau) - f(t) \in V
\]

for every \( t \in \mathbb{R} \).

Now for each \( s_n \), we can find \( \tau_n \) and \( \sigma_n \) such that \( s_n = \tau_n + \sigma_n \) with \( \tau_n \) a \( V \)-translation number of \( f \) and \( \sigma_n \in \{0, \varepsilon\} \) (it suffices to take \( \tau_n \in [s_n - \varepsilon, s_n] \) and then \( \sigma_n = s_n - \tau_n \)).

As \( f \) is uniformly continuous on \( \mathbb{R} \) (cf. [1, 2]), there exists \( \delta = \delta(\varepsilon) \) such that

\[
f(t') - f(t'') \in V
\]

for all \( t', t'' \), \( |t' - t''| < 2\delta \).

Also \( 0 \leq \sigma_n \leq \varepsilon \) for every \( n \); we can then subtract from \( (\sigma_n)^{m}_{n=1} \), a convergent subsequence \( (\sigma_{nk})^{m}_{k=1} \), by the Bolzano-Weierstrass theorem.

Let \( \sigma = \lim_{k \to \infty} \sigma_{nk} \), with \( 0 \leq \sigma \leq \varepsilon \).

Now consider the subsequence \( (\sigma_{nk})^{m}_{k=1} \) with

\[
\sigma - \delta < \sigma_{nk} < \sigma + \delta, \quad k = 1, 2, ...
\]

and let \( (s_{nk})^{m}_{k=1} \) be the corresponding subsequence where

\[
s_{nk} = \tau_{nk} + \sigma_{nk}, \quad k = 1, 2, ...
\]

Let us prove the relation

\[
f(t + s_{nk}) - f(t + s_{nj}) \in U
\]

for all \( t \in \mathbb{R} \).

For this, write

\[
f(t + s_{nk}) - f(t + s_{nj}) = f(t + \tau_{nk} + \sigma_{nk}) - f(t + \sigma_{nk})
+ f(t + \sigma_{nk}) - f(t + \sigma_{nj})
+ f(t + \sigma_{nj}) - f(t + \tau_{nj} + \sigma_{nj}).
\]

Because \( \tau_{nk} \) and \( \tau_{nj} \) are \( V \)-translation numbers of \( f \), we shall get
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\[ f(t + \tau_n - \sigma_n) - f(t + \sigma_n) \in V, \text{ for every } t \in \mathbb{R} \]
\[ f(t + \tau_j - \sigma_j) - f(t + \sigma_j) \in V, \text{ for every } t \in \mathbb{R}. \]

(2.3)

On the other hand
\[ |(t + \sigma_n) - (t + \sigma_j)| = |\sigma_n - \sigma_j| < 2\delta; \]
therefore, by using relation (2.1), we get
\[ f(t + \sigma_n) - f(t + \sigma_j) \in V, \text{ for every } t \in \mathbb{R}. \]

(2.4)

Finally we can deduce (2.2) from (2.3) and (2.4). The theorem is proved by taking
\[ s'_n = s_{n_k}, \quad k = 1, 2, \ldots \]

\[ \square \]

3. APPLICATIONS

Let \( E \) be a perfect Frechet space and \( A \) a closed linear operator with domain \( D(A) \) dense in \( E \). Suppose \( A \) generates a strongly continuous one-parameter group \( T(t), t \in \mathbb{R} \).

Consider in such \( E \) the differential equation
\[ x'(t) = Ax(t), \quad t \in \mathbb{R}. \]

THEOREM 2. Assume for every semi-norm \( p \in \mathbb{Q} \), there exists a semi-norm \( q \in \mathbb{Q} \) such that
\[ p(T(t)u) \leq q(u) \]
for every \( u \in E \) and \( t \in \mathbb{R} \).

Then every solution \( x(t) \) of (3.1) such that \( \{x'(t); t \in \mathbb{R}\} \) is relatively compact in \( E \) is a.p.

PROOF. Let \( x(t) \) be such a solution; we can write \( x(t) = T(t)x(0), t \in \mathbb{R} \); by the property on \( T(t) \), \( x(t) \) is obviously bounded.

Consider a given real sequence \( (s'_n)_{n=1}^\infty \); we can extract a subsequence \( (s_n)_{n=1}^\infty \) such that \( (x'(s_n))_{n=1}^\infty \) is a Cauchy sequence in \( E \), for \( \{x'(t); t \in \mathbb{R}\} \) is relatively compact in \( E \). We have
\[ x'(t + s_n) = Ax(t + s_n) \]
\[ = AT(t + s_n)x(0) \]
\[ = AT(t)x(s_n) \]
\[ = T(t)Ax(s_n) \]
\[ = T(t)x'(s_n) \]
for every \( n \) and every \( t \in \mathbb{R} \). Therefore
\[ x'(t + s_n) - x'(t + s_m) = T(t)[x'(s_n) - x'(s_m)] \]
for every \( n, m \) and \( t \in \mathbb{R} \).

Take now any \( p \in \mathbb{Q} \); then there exists \( q \in \mathbb{Q} \) such that
\[ p[x'(t + s_n) - x'(t + s_m)] \leq q[x'(s_n) - x'(s_m)] \]
for every \( t \in \mathbb{R} \); which shows \( x'(t) \) is a.p. by Bochner's criteria. As \( E \) is a perfect Frechet space, the conclusion is immediate.
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