FUNCTORIAL PROPERTIES OF THE LATTICE OF FUNCTIONAL SEMI-NORMS

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ABSTRACT. Given a measureable transformation between measure spaces, we determine when such gives rise to a mapping between the corresponding lattice of function semi-norms. We further determine when this mappings preserves norms and observe that it does preserve certain other important properties. We next establish a functorial connection between measure spaces and lattice. Finally, we show that the above lattice mapping does not commute with the associate construction.

KEY WORDS AND PHRASES: Function semi-norm, associate semi-norm, lattice of semi-norms, measure-preserving transformation, semi-norm preserving, associate preserving, lattice subhomomorphism, category, functor.


1. INTRODUCTION.

Let \((X, S, \mu)\) be a sigma-finite measure space and \(M^+(\mu)\) the space of \([0,\infty]\)-valued \(\mu\)-measurable functions on \(X\). Contrary to conventional practice, it will not be convenient to identify two functions in \(M^+(\mu)\) which are equal \(\mu\)-a.e. Accordingly, let \(Z(\mu)\) denote the \(\mu\)-null function in \(M^+(\mu)\). Thus, \(Z(\mu)\) is the null equivalence class in \(M^+(\mu)\) of the zero function on \(X\). In this setting, a (function) semi-norm on \(M^+(\mu)\) is a mapping \(\rho: M^+(\mu) \rightarrow [0,\infty]\) having the following properties. Let \(c>0\), and \(f, g \in M^+(\mu)\).

Then:
(1) \(f-g \in Z(\mu)\) implies \(\rho(f) = \rho(g)\).
(2) \(f \in Z(\mu)\) implies \(\rho(f) = 0\).
(3) \(\rho(cf) = c\rho(f)\).
(4) \(\rho(f+g) \leq \rho(f) + \rho(g)\).
(5) \(f \leq g \mu\)-a.e. implies \(\rho(f) \leq \rho(g)\).

The semi-norm \(\rho(f) = 0\) implies \(f \in Z(\mu)\). Let \(P(\mu)\) denote the set of all semi-norms and \(P^+_0(\mu)\) the subset of all norms (never empty).

Observe that \(P(\mu)\) is canonically partially ordered by:

\[ \rho_1 \leq \rho_2 \text{ if } \rho_1(f) \leq \rho_2(f), \quad f \in M^+(\mu). \]
It is well-known that, relative to this ordering, \( P(\mu) \) is a complete lattice with sup and inf given by

\[
(\rho_1 \vee \rho_2)(f) = \sup(\rho_1(f), \rho_2(f)),
\]

and

\[
(\rho_1 \wedge \rho_2)(f) = \inf\{\rho_1(f_1) + \rho_2(f_2): f_1, f_2 \in M^+(\mu), f_1 + f_2 = f, \mu-a.e.\}
\]

(See sections 3 and 4 of [3] for the sup and inf of arbitrary families in \( P(\mu) \).)

Now let \((Y,T,\nu)\) be another sigma-finite measure space and \( \phi:X \rightarrow Y \) a measurable transformation. For such \( \phi \), we obtain a mapping \( \phi^0:M^+(\nu) \rightarrow M^+(\mu) \) defined by \( \phi^0(g) = g\phi \). This in turn yields a mapping \( \phi^0 \circ \rho^0 \) from \( P(\mu) \) into the \([0,\infty]\)-valued functions on \( M^+(\nu) \). In general, \( \phi(\rho) = \rho \phi^0 \) is not a semi-norm. Moreover, if \( \rho \) is a norm, then \( \phi(\rho) \) may be a semi-norm which is not a norm. Thus, the first question we ask is: Under what conditions is \( \phi \) semi-norm-preserving? In section 2, we give necessary and sufficient conditions for this to be the case (2.2). The next question is: Under what additional conditions is \( \phi \) norm-preserving? In section 3, we give necessary and sufficient conditions for this to be the case (3.5). There are certain very important sublattices in the lattice of semi-norms which have been studied extensively (see [2,3]).

Also in section 3, we observe that all of these sublattices are preserved by \( \phi \) (3.7) when \( \phi \) is semi-norm-preserving. The previous results suggest there is a functorial connection between measure spaces and lattices. However, when \( \phi \) is semi-norm-preserving, \( \phi \) may not be a lattice homomorphism. Specifically, in general, "\( \phi \) of an infimum does not equal the infimum of the \( \phi \)'s". Despite this failing, \( \phi \) is a lattice "subhomomorphism" (4.3). With this notion of lattice morphism, we are able (in section 4) to establish the desired functorial connection. Finally, in section 5, we see that the mapping \( \phi \) and the assococonstruction \( \rho \circ \phi^0 \) are incompatible in general. For this purpose, recall that

\[
\rho^0(f) = \sup\{\int_X fg d\nu: \rho(g) \leq 1\}, f \in M^+(\mu)
\]

Also, let \( N(\mu) \) denote the space of \( \mu \)-null subsets of \( X \) (similarly for \( \nu \)).

2. SEMI-NORM PRESERVATION.

Before investigating the conditions under which \( \phi \) preserves semi-norms, let us see first that it does not have this property in general.

2.1 Example. Let \( X = Y = \{a,b\} \) with \( \mu \) and \( \nu \) defined as follows: \( \mu(\{a\}) = \mu(\{b\}) = \nu(\{a\}) = 1 \) and \( \nu(\{b\}) = 0 \). Let \( \phi \) be the identity mapping. Then \( \{b\} \in N(\nu) \), while \( \{b\} = \phi^-1(\{b\}) \notin N(\mu) \).

Let \( \rho \) be the \( L^1 \)-norm in \( P_0(\mu) \), i.e.,

\[
\rho(f) = \|f\|_1 = f(a) + f(b), f \in M^+(\mu)
\]

The function \( g \) on \( Y \) defined by \( g(a) = 0, g(b) = 1 \), is \( \nu \)-null. However, \( g\phi \) is not \( \mu \)-null, i.e. \( \phi^0(Z(\nu)) \notin Z(\mu) \). Thus, \( \phi(\rho)(g) = \rho(g\phi) \neq 0 \), i.e. \( \phi(\rho) \) is not constant on null equivalence classes in \( M^+(\nu) \).

2.2 Theorem. The following are equivalent:

(i) \( \phi:P(\mu) \rightarrow P(\nu) \)

(ii) \( \phi^-1(N(\nu)) \subseteq N(\mu) \)

(iii) \( \phi^0(Z(\mu)) \subseteq Z(\nu) \)
Proof. (ii) implies (i): Let \( g_1, g_2 \in \mathcal{M}^+(\nu) \) be such that \( g_1 \leq g_2, \nu\text{-a.e.} \). Then

\[
\{x \in X : g_1(x) \leq g_2(x)\} = \phi^{-1}(\{y \in Y : g_1(y) \leq g_2(y)\}).
\]

Since the set in the right parentheses is \( \nu\)-null, it follows from (ii) that its inverse image under \( \phi \) is \( \nu\)-null, i.e. \( g_1 \leq g_2, \mu\text{-a.e.} \). Hence, for \( \rho \in \mathcal{P}(\mu) \), we have

\[
\rho(\phi(g_1)) = \rho(g_1) \leq \rho(g_2) = \phi(\rho)(g_2),
\]

i.e. \( \phi(\rho) \) satisfies (5) of \( \dagger \). This also proves (i) of \( \dagger \). The remaining properties (2), (3), (4) of \( \dagger \) are easily verified. Therefore, \( \Phi(\rho) \in \mathcal{P}(\nu) \), for all \( \rho \in \mathcal{P}(\mu) \).

(iii) implies (ii): Let \( E \) be an element of \( \mathcal{N}(\nu) \). The characteristic function \( X_E \) is then in \( \mathcal{Z}(\nu) \), i.e. \( \Phi(X_E) \in \mathcal{Z}(\nu) \) by (iii). We then have

\[
X_E = \Phi(X_E) = X_{\phi^{-1}(E)}
\]

so that \( \phi^{-1}(E) \in \mathcal{Z}(\mu) \), i.e. \( \phi^{-1}(E) \in \mathcal{N}(\mu) \).

(i) implies (iii): Suppose (iii) is false. Then there exists \( f \in \Phi(\mathcal{Z}(\nu)) \) such that \( f \) is not in \( \mathcal{Z}(\mu) \). Let \( g \in \mathcal{Z}(\nu) \) be such that \( f = g \phi \). If \( \rho \in \mathcal{P}_0(\mu) \), then by (i) we must have \( \rho(\phi(g)) = \rho(f) = 0 \). This contradicts the fact that \( \rho \) is a norm.

2.3 Remarks. Observe that (iii) of the theorem says that \( \Phi \) essentially sends the zero-class in \( \mathcal{M}^+(\nu) \) to the zero-class in \( \mathcal{M}^+(\mu) \) because, modulo nullity, \( \Phi(\mathcal{Z}(\nu)) \supseteq \mathcal{Z}(\mu) \). Specifically, if \( f \in \mathcal{Z}(\mu) \), then \( \Phi(g) = f \), \( \mu\)-a.e., for \( g \) the zero function on \( Y \).

2.4 Definition. The measurable transformation \( \phi : X \rightarrow Y \) is semi-norm-preserving if the conditions of 2.2 hold.

3. PROPERTY PRESERVATION.

The natural next question to ask about \( \phi \) is the following: Under what conditions does it preserve norms? The answer to this question is somewhat complicated because of some measure-theoretic technicalities. These (together with some additional notation) are necessitated by the fact that \( \phi \) need not preserve measurable sets, i.e. it may not be bimeasurable.

Let \( \overline{\nu} \) denote the completion of \( \nu \) and \( \overline{T} \) its domain [1]. Let \( \nu^* \) (resp. \( \nu^* \)) denote the outer (resp. inner) measure derived from \( \nu \). Also let

\[
N_\phi(\mu) = \{E \in \mathcal{N}(\mu) : \phi^{-1}(\phi(E)) = E\}.
\]

In general, \( N_\phi(\mu) \) is a proper subset of \( \mathcal{N}(\mu) \). However:

3.1 Lemma. The transformation \( \phi \) is semi-norm-preserving, (i.e. \( \phi^{-1}(N(\nu)) \subseteq N(\mu) \)) if and only if \( \phi^{-1}(N(\nu)) \subseteq N_\phi(\mu) \).

Proof. The elements of \( \phi^{-1}(N(\nu)) \) automatically have the extra property.

For any semi-norm \( \rho \), define

\[
K(\rho) = \{f \in M^+(\mu) : \rho(f) = 0\}.
\]

Of course, \( K(\rho) \supseteq Z(\mu) \) in general.

3.2 Lemma. Suppose \( \phi \) is semi-norm preserving. If \( \rho \in \mathcal{P}(\mu) \), then

\[
K(\phi(\rho)) = (\phi^0)^{-1}(K(\rho)).
\]

Proof. Straightforward.

We then have the following answer to our question:

3.3 Proposition. Suppose \( \phi \) is semi-norm preserving and \( \rho \) is a norm in \( \mathcal{P}(\mu) \). Then \( \Phi(\rho) \) is a norm if and only if \( (\phi^0)^{-1}(Z(\mu)) = Z(\nu) \).

Proof. Apply 2.2 and 3.2.

In order to obtain an answer analogous to 2.2 in terms of \( \phi \) itself, we first require the following.
3.4 Lemma. Let \( C = Y - \phi(X) \) (set difference). If \( \rho \in P(\mu) \) and \( \Phi(\rho) \) is a norm, then \( \nu_*(C) = 0 \), i.e. \( \phi \) is \( \nu_* \)-essentially onto.

Proof. If not, there exists \( E \in \mathcal{T} \) such that \( E \subseteq C \) and \( \nu(E) > 0 \). Then for \( g = \chi_E \), we have \( g \in Z(\mu) \), so that \( \Phi(\rho)(g) = 0 \), while \( g \notin Z(\nu) \).

3.5 Theorem. Suppose \( \phi \) is semi-norm preserving, \( \rho \) is a norm in \( P(\mu) \) and \( \nu_*(C) = 0 \).

If \( \phi(X) \in \mathcal{T} \), then the following are equivalent:

(i) \( \Phi(\rho) \) is a norm.

(ii) \( \Phi(\mu) \subseteq \Phi^{-1}(N(\nu_*)) \).

(iii) \( (\phi)^{-1}(Z(\mu)) = Z(\nu) \) (recall 2.2, 2.3).

Proof. (i) is equivalent to (iii) by 3.3.

(ii) implies (iii): Let \( E \in \Phi(\mu) \), so that \( \mu(E) = 0 \) and \( \phi^{-1}(\Phi(E)) = E \). Then \( \chi_E \in Z(\mu) \) and \( \chi_E \Phi(E) = \chi_E \). Let \( F \in \mathcal{T} \) be such that \( F \subseteq \Phi(E) \). Then \( \chi_F \leq \chi_E \Phi(E) \) and \( \chi_F \Phi(E) \leq \chi_F \Phi(E) \Phi(E) = \chi_E \). Hence, \( \chi_E \Phi(E) \subseteq Z(\mu) \), i.e. \( \phi^{-1}(\chi_F) \in Z(\mu) \). This implies that \( \chi_F \in (\phi^{-1}(Z(\mu)))^{-1} \), \( Z(\nu) \), i.e. \( \nu(\phi(E)) = 0 \). Consequently, \( \nu_*(\phi(E)) = 0 \), so that \( \phi(E) \in N(\nu_*) \). (ii) implies (i): Let \( g \in M^+(\nu) \) be such that \( \Phi(\rho)(g) = 0 \). Then \( \rho(g) = 0 \), so that \( g \in Z(\nu) \) \((\rho \text{ is a norm})\). Since

\[ \phi^{-1}(\text{supp}(g)) = \text{supp}(g\phi), \]

it follows that \( \mu(\phi^{-1}(\text{supp}(g))) = 0 \), i.e. \( \phi^{-1}(\text{supp}(g)) \in \Phi(\mu) \). Let

\[ G_r = \text{supp}(g) \cap \Phi(X), \quad G_c = \text{supp}(g) \cap C, \]

observing that \( \Phi(X) \) and \( C \) belong to \( \mathcal{T} \). Then \( \Phi(X) \cap \Phi(C) = G_r \cup G_c \) (disjoint) and

\[ \phi^{-1}(\text{supp}(g)) = \phi^{-1}(G_r) \in \Phi^{-1}(N(\nu_*)), \]

by condition (ii), so that \( \nu_*(G_r) = 0 \). On the other hand,

\[ v_*(G_r) = \bar{v}(G_r) = \nu_*(G_r) = 0 \] \[ [1, \text{p. 60}], \]

Therefore,

\[ v(\text{supp}(g)) \leq \nu_*(G_r) + v_*(G_c) = 0 \] \[ [1, \text{p. 61}], \]

so that \( g \in Z(\nu) \), i.e. \( \Phi(\rho) \) is a norm.

3.6 Corollary. Suppose \( \phi \) is semi-norm-preserving and \( \rho \) is an norm in \( P(\mu) \). If \( \phi \) is bimeasurable and maps \( X \nu \)-essentially onto \( Y \), then (i) and (iii) of the theorem are equivalent to (ii'): \( N_\phi(\mu) = \phi^{-1}(N(\nu)) \).

We next consider our question in the context of the subsets of \( P(\mu) \) introduced in section 2 of [3]. Here the answers are the best possible. The subsets consist of those norms having the Riesz-Fisher (R), weak (W) or strong (S) Fatou property, those satisfying the infinite triangle inequality (I) and those which are of absolutely continuous norm (A) (see [2, 3, 4]).

3.7 Theorem. For the following, let \( B \) denote either R, I, W, S or A. If \( \phi \) is semi-norm-preserving, then \( \phi \) preserves the property defining \( B \), i.e. \( \phi: B(\mu) \rightarrow B(\nu) \).

Proof. The proof for each choice of \( B \) is more-or-less straightforward. Therefore, we leave the details to the interested reader - after remarking that 3.2 is required in proving the theorem for the case \( B = R \).

4. THE FUNCTOR.

In this section we investigate the categorical connection between measurable transformations and lattices of semi-norms. As the next example shows, if \( \phi \) is semi-norm-preserving, the corresponding morphism \( \phi \) may not be a lattice homomorphism.
4.1 **Example.** Let $X = N$ the set of all positive integers, and $Y = \{y\}$ with $v(Y) = 1$. Define $\phi(x) = y$, $x \in X$. Then $\phi$ is a semi-norm-preserving measurable transformation and we have $\Phi : P(\mu) \to P(\nu)$ as in Section 2. Define
\[ \rho_1(f) = \sum_{n=1}^{\infty} f(n)/2^n \]
and
\[ \rho_2(f) = \lim_{n\to\infty} \sup(f(n)) + \sup(f(n)), f \in M^+(\mu). \]
Let $g$ be the function equal to 1 on $Y$, so that $g \in M^+(\nu)$. We leave to the reader the verification that
\[ \left[ \Phi(\rho_1 \wedge \rho_2) \right] (g) = 1, \]
while
\[ \left[ \Phi(\rho_1 \vee \rho_2) \right] (g) \leq 1. \]
Thus, $\Phi(\rho_1 \wedge \rho_2) \neq \Phi(\rho_1) \wedge \Phi(\rho_2)$ in general.

Despite this failing, $\Phi$ does have suitable lattice morphism properties.

4.2 **Lemma.** If $\rho_1, \rho_2 \in P(\mu)$, then $\Phi(\rho_1 \vee \rho_2) = \Phi(\rho_1) \vee \Phi(\rho_2)$ and $\Phi(\rho_1 \wedge \rho_2) \leq \Phi(\rho_1) \wedge \Phi(\rho_2)$ in general.

4.3 **Definition.** Any mapping between lattices having the properties exhibited by $\Phi$ in 4.2 will be called a lattice subhomomorphism.

We are now ready to define a functor. On the one hand, consider all sigma-finite measure spaces as the objects and semi-norm-preserving, measurable transformations as the morphisms. These form a category which we denote by $X$. On the other hand, consider all lattices as the objects and lattice subhomomorphisms as the morphisms. These form a category which we denote by $P$. By the results of section 2, we obtain a "mapping"
\[ F : X \to P \]
determined by
\[ F(X, S, \mu) = P(\mu), (X, S, \mu) \in \text{Obj}(X), \]
and
\[ F(\Phi) = \Phi, \Phi \in \text{Mor}((X, S, \mu), (Y, T, \nu)). \]

We leave to the reader the task of verifying the $F$ is in fact a functor.

5. ASSOCIATE PRESERVATION.

Our final concern is the question of whether $\Phi$ preserves associates. We shall see in the next examples that $\Phi(\rho')$ and $\Phi(\rho)'$ are not even comparable in general.

5.1 **Example.** Let $X, Y, \nu, \phi$ be as in 4.1. Denote the respective characteristic functions of $X, Y$ by $f, g$. Let $\rho$ be the norm in $P(\mu)$ given by
\[ \rho(h) = \sum_{1}^{\infty} h(n)/2^n, \quad h \in M^+(\mu). \]
Then $\rho(f) = 1$ and
\[ \Phi(\rho')(g) = \sup \{|h(y)| : \rho(\phi) \leq 1\} \]
\[ = \sup \{|h(y)| : h(y) \rho(f) \leq 1\} \]
\[ = \rho(f) - 1 \]
\[ = 1. \]

On the other hand,
\[ \Phi(\rho')(g) = \sup \{\sum_{1}^{\infty} |h(n)| : \rho(h) \leq 1\} \]
\[ \leq \sum_{1}^{\infty} |f(n)| \]
Thus, $\Phi(\rho') \neq \Phi(\rho)'$, in general.
5.2 Example. Now let $X = \{x\}$, $Y = \mathbb{N}$ with $\mu(X) = 1$. Define $\phi(x) = 1$, so that $\phi$ is semi-norm-preserving. Let $h$ denote the characteristic function of $Y$ and define
\[
g(y) = 0, \quad y = 1
\]
\[
= 1, \quad y \neq 1.
\]
Let $\rho$ be the norm in $P(\mu)$ given by $\rho(f) = f(x)$. Then
\[
\phi'(g) = g(1) \sup\{|f(1)| : \rho(f) \leq 1}\]
\[
= 0.
\]
On the other hand,
\[
\phi'(g) = \sup\{|f(n)| : \rho(f) \leq 1\}
\]
\[
\leq \|f\|_2
\]
\[
= \infty.
\]
Thus, $\phi'(g) \neq \phi'(g)$, in general

5.3 Remarks. It is possible to find non-trivial conditions on $\phi$ which will at least guarantee a comparison of $\phi'(g)$ and $\phi'(g)$. However, the conditions we have in mind are not far from requiring that $\phi$ be an essential measure isomorphism (need not be essentially one-one). Thus, the strength of the hypothesis, combined with the weakness of the conclusion (namely, $\phi'(g) \geq \phi'(g)$), provide little motivation for presenting the details here.

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