ABOUT THE DEFECTS OF CURVES HOLOMORPHIC IN THE HALF PLANE

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ABSTRACT. A study is made on the defects of curves holomorphic in the half plane. Several results are proved.

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1. INTRODUCTION.

Let \( f(z) \) be a meromorphic function in the half plane \( \{ \text{Im} z > 0 \} \) and \( f(z) \) is not equal to a constant identically. We denote by \( n(r,f), r > 1 \), the number of poles of \( f(z) \) lying inside the set \( \{ z: |z-\frac{1}{2}| < \frac{r}{2}, |z| > 1 \} \), \( 1 < r < \infty \). We put \([1,2,3]\)

\[
N(r,f) = \int_{1}^{r} \frac{n(t,f)}{t^2} \, dt + \sum_{\rho_k^\epsilon \in D(r,1)} \frac{\sin \psi_k}{\rho_k^\epsilon} - \frac{1}{r^2},
\]

where \( \rho_k^\epsilon \) are the poles of the function \( f(z) \).

\[
N(r,\frac{1}{f-a}) = N(r,a),
\]

\[
m(r,f) = \frac{1}{\pi} \int_{\arcsin \frac{1}{r}}^{\frac{\pi}{2}} \ln^+ |f(re^{i\theta})| \frac{d\theta}{\sin^2 \theta},
\]

\[
m(r,a,f) = m(r,a),
\]

\[
T(r,f) = m(r,f) + N(r,f).
\]

When the half circle \( \{ |z| = 1, \text{Im} z > 0 \} \) does not contain neither zeros nor poles of the function \( f(z) - a \), we get the following equality [1]:

\[
\int_{\arcsin \frac{1}{r}}^{\frac{\pi}{2}} \ln |f(re^{i\theta}) - a| \frac{d\theta}{\sin^2 \theta} + \Theta(r,1,f-a) = \int_{\arcsin \frac{1}{r}}^{\frac{\pi}{2}} \ln |f(re^{i\theta}) - a| \frac{d\theta}{\sin^2 \theta} + \Theta(r,1,f-a) \tag{1.1}
\]

where

\[
\Theta(r,1,f-a) = \frac{1}{\pi} \int_{\arcsin \frac{1}{r}}^{\frac{\pi}{2}} \ln |f(re^{i\theta}) - a| \frac{d\theta}{\sin^2 \theta}.
\]
The order $\rho_T(f)$ of the function $f(z)$ in the sense of TSUJI is called the number

$$\rho_T(f) = \lim_{r \to \infty} \frac{\ln T(r,f)}{\ln r}.$$

Similarly, the lower order $\lambda_T(f)$ of $f(z)$ is defined by the quantity

$$\lambda_T(f) = \lim_{r \to \infty} \frac{\ln \overline{T}(r,f)}{\ln r}.$$

The quantity

$$\delta_T(a,f) = \delta_T(a) = \lim_{r \to \infty} \frac{\bar{m}(r,a)}{T(r,f)} = 1 - \lim_{r \to \infty} \frac{\bar{N}(r,a)}{T(r,f)}$$

is called the deficiency of $f(z)$ at the point $a$. The value $a$ of $f(z)$, for which $\delta_T(a) > 0$, is called a deficiency of $f(z)$. For a meromorphic function of order $\rho_T$ and lower order $\lambda_T$ in the half $z$-plane $\{\text{Im}z > 0\}$, we put $E_T(f) = \{a: \delta_T(a,f) > 0\}$. The set $E_T(f)$ is called the set of deficient values of $f(z)$ in the sense of TSUJI.

2. RESULTS ANALOGOUS TO TSUJI'S WORK.

We will derive analogue of TSUJI'S work for curves holomorphic in the half plane.

Let $C^p$ be a $p$-dimensional complex unitary space and $\bar{a}$ - vectors from $C^p$. The vector

$$\vec{g}(z) = \{g_1(z), g_2(z), \ldots, g_p(z)\}$$

is called $p$-dimensional entire curve of the complex parameter $z$, where the function

$$(g_n(z))_{n=1}^p$$

are linearly independent entire functions without common zeroes. The scalar product:

$$\langle \vec{g}(z), \vec{a} \rangle = \sum_{k=1}^{p} g_k(z) a_k$$

is called an ordinary entire function of the parameter $z$.

Definition: The finite ($\geq p$) or infinite system $A$ of vectors is called admissible, if any $p$-vectors of the system are linearly independent.

For $n = 2$, we shall get ordinary meromorphic functions. In this case the characteristics of $T^*$ and $\overline{T}$ coincide.

The quantity $n^*(r,\vec{a},\vec{c})$ will denote the number of zeroes of the holomorphic function $\langle \vec{g}(z), \vec{a} \rangle$ in the region $\{z: |z - \frac{1}{2}| < \frac{r}{2}, |z| > 1\}$, where $\vec{a} \in A$, the admissible system of vectors.

We put

$$N^*(r,\vec{a},\vec{c}) = \int_{1}^{r} n^*(t,\vec{a},\vec{c}) \frac{dt}{t^2}, \quad 1 < r < \infty.$$

We define the function $m^*(r,\vec{a},\vec{c})$ and $T^*(r,\vec{c})$ by the quantities:

$$m^*(r,\vec{a},\vec{c}) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\pi}{\arcsin \frac{1}{r}} \ln^+ \frac{|\vec{g}(re^{i\theta})|}{|\vec{g}(re^{i\theta}\vec{a})|} \cdot \frac{d\theta}{\sin^2 \theta},$$

where $\vec{a} \in A$. The value $a$ of $f(z)$, for which $\delta_T(a) > 0$, is called a deficiency of $f(z)$. For a meromorphic function of order $\rho_T$ and lower order $\lambda_T$ in the half $z$-plane $\{\text{Im}z > 0\}$, we put $E_T(f) = \{a: \delta_T(a,f) > 0\}$. The set $E_T(f)$ is called the set of deficient values of $f(z)$ in the sense of TSUJI.
where
\[ ||\tilde{c}(z)|| = \sqrt{\sum_{i=1}^{p} |k_i(z)|^2}. \]

For all admissible system \( \mathbb{A} \subset \mathbb{C}^p \) of vectors we get the equality:
\[ M^*(r,\tilde{c}) = \mathbb{M}(r,\mathbb{A},\tilde{c}) + M^*(r,\mathbb{A},\tilde{c}) + O(1), \]
when \( r \rightarrow \infty \).

The order \( \rho_T \) and the lower order \( \lambda_T \) of the entire curve \( \hat{c}(z) \) may be defined as before.

For a meromorphic function in the half plane the following assertions are true:

**THEOREM A.** [2] Let \( 0 < \rho < \infty \) and \( M \) be an arbitrary, not more than countable, set of points in the extended complex plane. Then there exists a meromorphic function in the half plane \( \{\text{Im} z > 0\} \) of order \( \rho_T \), for which the set \( E_T(f) \) of deficient values coincides with \( M \).

**THEOREM B.** [2] Let \( \{n_\nu\} \) be a sequence of positive number which satisfies the condition \( \sum_{\nu=1}^{\infty} n_\nu = 1 \) and let \( \{a_k\} \) be an arbitrary sequence of different complex numbers. For any \( \lambda, \ 0 < \lambda < \infty \), there exists a meromorphic function in the half plane \( \{\text{Im} z > 0\} \) of lower order \( \lambda \) such that \( \delta_T(a_k,f) \geq K_1(\rho,\theta_1)n_k^3 \), for \( \lambda \leq 1 \),
\[ \delta_T(a_k,f) \geq K_1(\rho,\theta_1)n_k^3 \), for \( \lambda > 1 \),

where \( K_1(\rho,\theta_1) = \min \left\{ \frac{\rho^3}{36(\rho+1)}, \frac{\rho^3}{18(\rho+1)} \sin^{\rho-1}\theta_1 \right\} \).

**THEOREM C.** [2] For any \( \lambda, \ 0 < \lambda < \infty \), there exists a meromorphic function in the half plane \( \{\text{Im} z > 0\} \) of finite lower order \( \lambda \) such that the series \( \sum \delta_T^2(a_k,f) \) converges for each \( a < \frac{1}{3} \).

**THEOREM D.** [3] Let \( \{w_\nu\} \) be an arbitrary sequence of complex number and \( \{\delta_\nu\}_1^{N} \) be a sequence of positive number \( (N \leq \infty) \) satisfying the relationship \( \sum_{\nu=1}^{\infty} \delta_\nu \leq 1 \).

Then there exists a meromorphic function in the half plane \( \{\text{Im} z > 0\} \) of finite order \( \lambda, \ 0 < \lambda < \infty \), such that
\[ \delta_T(w_\nu,f) = \delta_\nu, \ w_\nu \in \{w_\nu\} \) and \( \delta_T(w,f) = 0, \) if \( \nu \notin \{w_\nu\} \).

**MAIN RESULTS.**

In this article we shall consider the following assertions:

We denote by \( \mathbb{A}(w) \) the fixed admissible system \( \mathbb{A} \) of vectors, if the coordinates of the vector \( \mathbb{A} \) of the system depend on a complex parameter \( w \). We shall denote the the vectors of the admissible system \( \mathbb{A}(w) \) of vectors by \( \mathbb{A}(w) \).

**THEOREM.** Let \( f(z) \) be an arbitrary meromorphic function in the half plane. It
is possible to find a $p$-dimensional holomorphic curve $\tilde{G}(z)$ ($p \geq 2$) and an admissible system $A(\omega)$ of vectors such that ($r > 1$)

$$T^*(r, \tilde{G}) = (p - 1)\overline{T}(r, f) + O(1),$$

$$\nu^*(r, \tilde{a}(\omega), \tilde{\omega}) = (p - 1)\overline{\nu}(r, \omega, f) + O(1),$$

$$\delta_T(\tilde{a}(\omega), \tilde{\omega}) = \delta_T(\omega, \tilde{\omega}), \quad \Delta_T(\tilde{a}(\omega), \tilde{\omega}) = \Delta_T(\omega, f).$$

**COROLLARY 1.** Let $0 < \rho < \infty$ and $M$ be an arbitrary, not more than countable, set of vectors of the admissible system of vectors. Then there exists a $p$-dimensional ($p \geq 2$) curve $\tilde{G}(z)$ holomorphic in the half plane $\{ \text{Im} z > 0 \}$ of order $\rho_T$, whose set of deficient values $E_T(f)$ coincides with $M$.

**COROLLARY 2.** For any $\lambda$, $0 < \lambda < \infty$ and $p > 2$, there exists a $p$-dimensional curve $\tilde{G}_\lambda(z)$ holomorphic in the half plane $\{ \text{Im} z > 0 \}$ of lower order $\lambda$ and an admissible system $A(\omega)$ of vectors containing the sequence of distinct vectors:

$$\tilde{a}_k = \tilde{a}(\omega_k) \in A(\omega), \quad k = 1, 2, \ldots$$

such that ($q = [\lambda] + 1$)

$$\delta_T(\tilde{a}_k, \tilde{\omega}) \geq K_1(\rho, \theta) \eta_k^3, \quad \text{for } \lambda \leq 1,$$

$$\delta_T(\tilde{a}_k, \tilde{\omega}) \geq K_1(\rho, \theta) \eta_k^3, \quad \text{for } \lambda > 1,$$

where

$$K_1(\rho, \theta) = \min \left\{ \frac{\rho^3}{36(\rho + 1)}, \frac{\rho^3}{18(\rho + 1)} \sin^2 \theta \right\}.$$

**COROLLARY 3.** For any $\lambda$, $0 < \lambda < \infty$, $p > 2$, there exists a $p$-dimensional curve $\tilde{G}_\lambda(z)$ holomorphic in the half plane $\{ \text{Im} z > 0 \}$ of finite lower order $\lambda$ and an admissible system $A(\omega)$ of vectors such that the series $\sum_{a \in A(\omega)} \frac{\delta_T(\tilde{a}(\omega), \tilde{\omega})}{a}$ diverges for $a < \frac{1}{3}$.

**COROLLARY 4.** Let $\{w\}$ be an arbitrary sequence of complex number and $\{\delta_\nu\}^N_{\nu=1}$ be a sequence of positive number ($N < \infty$) satisfying the condition $\sum_{\nu=1}^N \delta_\nu = 1$. Then for any $\lambda$, $0 < \lambda < \infty$, $p > 2$, there exists a $p$-dimensional curve $\tilde{G}_\lambda(z)$ holomorphic in the half plane $\{ \text{Im} z > 0 \}$ such that $\delta_T(\tilde{a}(\omega), \tilde{\omega}) = \delta_\nu$ and

$$\delta_T(\tilde{a}(\omega), \tilde{\omega}) = 0,$$  

if $\omega \in \{w\}$.

**PROOF OF THEOREM 1.** Let $f(z) = \frac{g_1(z)}{g_2(z)}$ (2.1)

be a meromorphic function in the half plane $\{ \text{Im} z > 0 \}$, where $g_1(z)$ and $g_2(z)$ are entire functions having no zeroes in common.

We note that the entire functions:

$$h_k(z) = g_1^{p-k-1}(z) g_2^k(z), \quad k = 0, 1, \ldots, p-1$$

are linearly independent in the field of complex number [5].
We consider the curve
\[ \mathcal{C}(z) = \{ h_0(z), h_1(z), \ldots, h_{p-1}(z) \} \tag{2.3} \]
holomorphic in the half plane \( \{ \text{Im } z > 0 \} \) and the system \( A, A = A(w) = \{ \mathbf{a}(w) \} \) where for any complex \( w \neq \infty \),
\[ \mathbf{a}(w) = (1, \ldots, (-1)^{k-1}(-1)^{p-1-k}, \ldots, (-1)^{p-1}w^{-n-1}) \].

If \( \{ \mathbf{a}(w_k) \}_{k=1}^{p-1} \), \( p \)-different vectors from the system \( A \), then for the determinant \( \Delta \) of the system we have
\[ |\Delta| = (-1)^{\frac{p(p-1)}{2}} \prod_{k=1}^{p-1} (\mathbf{w}_k - \mathbf{w}_1) \neq 0. \]

Thus \( A(w) \) is an admissible system of vectors. We shall investigate the defects of the curve \( \mathcal{C}(z) \) holomorphic in the half plane \( \{ \text{Im } z > 0 \} \) for the vectors \( \mathbf{a}(w) \) contained in this admissible system \( A(w) \).

We have
\[ \mathcal{C}(z) \mathbf{a}(w) = \sum_{k=0}^{p-1} g^{p-1-k}(-1)^k c^p \mathbf{g}_2(z) \mathbf{w}^k = (g_1(z) - g_2(z)w)^{p-1} = g_2^{p-1}(z)(f(z) - w)^{p-1}. \tag{2.4} \]

For \( z = re^{i\theta} \) we get
\[ \left| \mathcal{C}(re^{i\theta}) \mathbf{a}(w) \right| = \sqrt{\sum_{k=0}^{p-1} \frac{1}{|f(z)|^{2k}}} \leq \max \left( \frac{|f(z)|^{2(p-1)}}{|f(re^{i\theta}) - w|^{p-1}} \right). \tag{2.5} \]

Then
\[ \ln^+ \frac{1}{|f(re^{i\theta}) - w|} = \ln^+ \frac{1}{|f(re^{i\theta}) - w|^{p-1}} \leq \ln^+ \frac{\mathbf{a}(w)}{|\mathbf{a}(w)|} \max(1, |f(z)|^{p-1}) \leq \ln \frac{\mathcal{C}(re^{i\theta}) \mathbf{a}(w)}{|\mathcal{C}(re^{i\theta}) \mathbf{a}(w)|} \left( |\mathbf{a}(w)| > 1 \right). \tag{2.7} \]

From (2.6) and (2.7) we get
\[ \left| \mathcal{C}(re^{i\theta}) \mathbf{a}(w) \right| \leq \sqrt{p \left( |f(re^{i\theta}) + 1|^{p-1} \right) / |f(re^{i\theta}) - w|^{p-1}} \left| \mathbf{a}(w) \right|. \tag{2.8} \]

For the curve \( \mathcal{C}(z) \) holomorphic in the half plane defined by the relation (2.3) we get by (2.7) and (2.8)
\[ (p-1)m(r,w,f) \leq m^*(r, \mathbf{a}(w), \mathcal{C}) \leq (p-1)m(r,w,f) + C. \tag{2.9} \]

Since the characteristics \( m(r,f) \) consider the arguments of the poles, the characteristics \( n(r,f) \) may decrease, if we simply change its argument for fixed value of its modulus. Then from (2.4), (2.5), and (1.1), we get
\[(p - 1)\overline{\overline{N}}(r, w, f) + (p - 1)\overline{\overline{N}}(r, w, f) - (p - 1)\overline{\overline{N}}(r, w, f) = N^*(r, a(w), \mathcal{O}) . \tag{2.10}\]

\[(p - 1)\overline{\overline{N}}(r, w, f) = N^*(r, a(w), \mathcal{O}) \leq T^*(r, \mathcal{O}) + C.\]

From (2.9) and (2.10) we get
\[T^*(r, \mathcal{O}) = m^*(r, a(w), \mathcal{O}) + N^*(r, a(w), \mathcal{O}) \leq (p - 1)m(r, w, f) + C + (p - 1)\overline{\overline{N}}(r, w, f) \leq (p - 1)\overline{T}(r, f) + C .\]

If in (2.9), \(f(z) = g(z)\) is an analytic function, then we shall proceed in the same way considering \(g_1(z) = g(z)\) and \(g_2(z) = 1\) in (2.1), (2.2) and (2.3) with
\[\mathcal{O}(z) = \{ g^{p-1}(z), g^{p-2}(z), \ldots, 1 \} .\]

Corollary 1, 2, 3 follow from Theorem 1, and Theorem A, B, C, for meromorphic functions. Corollary 4 follows from Theorem 1 and Theorem D for analytic function defined in ([3], 131-135).

REFERENCES

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