ABSTRACT. For $\alpha > 0$, let $B(\alpha)$ be the class of regular normalized Bazilević functions defined in the unit disc. Choosing the associated starlike function $g(z) = z$ gives a proper subclass $B_{1}(\alpha)$ of $B(\alpha)$. For $B(\alpha)$, correct growth estimates in terms of the area function are unknown. Several results in this direction are given for $B_{1}(1/2)$.

KEY WORDS AND PHRASES. Bazilevic functions, subclasses of $S$, functions whose derivative has positive real part, close-to-convex functions, coefficient and length-area estimates.

1980 AMS SUBJECT CLASSIFICATION CODE. 30C45.

1. INTRODUCTION.

Let $S$ be the class of regular, normalized, univalent functions with power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n}$$

for $z \in D$, where $D = \{ z : |z| < 1 \}$.

Denote $R, S^{*}, K$ and $B(\alpha)$ the subclasses of $S$ which are functions whose derivative has positive real part [8], starlike with respect to the origin [9, p.221], close-to-convex [6] and Bazilevic of type $\alpha$ [13] respectively. Following [13] we define $f \in B(\alpha), \alpha > 0$ to be the class of functions $f$, regular and normalized in $D$, such that, there exist $g \in S^{*}$ such that for $z \in D$,

$$\Re \frac{zf'(z)}{f(z)} > 0.$$  \hspace{1cm} (1.2)

Then if $g(z) = f(z)$, $B(0) = S^{*}$ and $B(1) = K$. Let $C(r)$ denote the closed curve which is the image of $D$ under the mapping $w = f(z)$, $L(r)$ be the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. For $f \in S^{*}$, it was shown [7] that, with $z = r e^{i\theta}$, $0 < r < 1$,

$$L(r) = O(1) \left( M(r) \log \frac{1}{1-r} \right) \text{ as } r \to 1,$$  \hspace{1cm} (1.3)

where $M(r) = \max |f(z)|$, and Hayman [4] gave an example to show that this estimate is best possible when $f$ is bounded. In [14] this result was extended to starlike func-
tions with $A(\tau) < A$ constant. A modification of this method also shows that for $f \in S^*$,

$$L(\tau) = O(1) \sqrt{A(\tau) \log \frac{1}{1-\tau}} \text{ as } \tau \to 1.$$  \hspace{1cm} (1.4)

Thomas [14] also showed that (1.3) holds for the class $K$ and for the class $B(\alpha)$, $0 < \alpha \leq 1$ [13]. It is apparently an open question that (1.4) is valid for $f \in K$ or $B(\alpha)$.

Pommerenke [11] showed that if $f \in S^*$, then for $n \geq 2$

$$n|a_n| \leq c_n A(1 - \frac{1}{n}),$$ \hspace{1cm} (1.5)

where $C$ is constant, and Noor [10] extended this to $B(\alpha)$ by showing that

$$n|a_n| \leq C M(1 - \frac{1}{n}).$$  \hspace{1cm} (1.6)

The question as to whether (1.5) is valid for $f \in K$ or $B(\alpha)$ is also apparently open.

In [12] the subclass $B_1(\alpha)$ of $B(\alpha)$ consisting of those functions in $B(\alpha)$ for which $g(z) = z$ was considered and sharp estimates for the modules of the coefficients $a_2, a_3,$ and $a_4$ were given. In [15] Thomas gave sharp estimates for the coefficients $a_n$ in (1.1) when $a = 1/N$, $N$ a positive integer.

In this paper we shall be concerned with the class $B_1(1/2)$ and will use the method of Clunie and Keogh [1] to establish (1.5) and hence (1.6) and the method of Thomas [14] to prove (1.4) and hence (1.3). The methods will in fact give results which are stronger for this subclass.

2. STATEMENTS OF MAIN RESULTS.

**THEOREM 1.** Let $f \in B_1(1/2)$ and be given by (1.1). Then

(i) $n|a_n| \leq o(1) + O(1) \sqrt{A(1 - \frac{1}{n})}$, as $n \to \infty$,

(ii) $L(\tau) = O((1)\sqrt{A(\tau) \log \frac{1}{1-\tau}}) \text{ as } \tau \to 1.$

We shall need the following:

**LEMMA 1.** Let $f \in B_1(1/2)$ and be given by (1.1). Define the function $F$ in $D$ by $F(z)^2 = f(z^2)$. Then

$$A(\tau, F) \leq \frac{1}{2(\pi-2)^2} A(\tau^2, f).$$

**PROOF.** For $z = re^{i\theta}$, $0 \leq r < 1$,

$$A(\tau, F) = \int_0^{2\pi} \int_0^\tau \left| F'(z) \right|^2 \rho d\rho d\theta$$

$$= \int_0^{2\pi} \int_0^\tau \left| zf'(z^2) \right|^2 \rho d\rho d\theta.$$

Now

$$\left| \frac{1}{f(z^2)} \right| \leq \frac{4}{(\pi-2)^2}$$ \hspace{1cm} [15] \hspace{1cm} and so using (1.1) we have

$$A(\tau, F) \leq \frac{4}{(\pi-2)^2} \int_0^{2\pi} \int_0^\tau \left| zf'(z^2) \right|^2 \rho d\rho d\theta$$

$$\leq \frac{\pi}{2(\pi-2)^2} \sum_{n=1}^{\infty} n|a_n|^2 r^{4n},$$ \hspace{1cm} (where $|a_1| = 1$)

$$= \frac{1}{2(\pi-2)^2} A(\tau^2, f).$$
LEMMA 2. For \( f \in S \)

(i) the map \( r \to (1-r)^b \frac{A(r)}{(1+r)^2} \) is decreasing on the interval \((0, 1)\),

(ii) \( A(i/r) < \frac{512}{r} A(r) \) for \( 0 < r < 1 \).

PROOF. Since

\[
A(r) = \int_0^{2\pi} |f'(z)|^2 \, r \, d\theta,
\]

we have

\[
r A'(r) = \int_0^{2\pi} |zf'(z)|^2 \, d\theta
\]

\[
\leq 2 \int_0^{2\pi} r |f'(z)|^2 \left| \frac{zf'(z)}{f'(z)} \right| \, \rho \, d\theta + 2 \int_0^{2\pi} r |f'(z)|^2 \, \rho \, d\theta.
\]

The classical distortion theorem for \( f \in S \) [3, p. 5] gives

\[
r A'(r) \leq 4r (r^2 + 2) \int_0^{2\pi} |f'(z)|^2 \, \rho \, d\theta + 2 \int_0^{2\pi} r |f'(z)|^2 \, \rho \, d\theta.
\]

Thus

\[
\frac{d}{dr} \left( \log A(r) \right) \leq \frac{d}{dr} \left( \log \left( \frac{r^2 (1+r)^2}{1-r^b} \right) \right)
\]

and part (i) of Lemma 2 is now obvious. Part (ii) follows immediately.

PROOF OF THEOREM 1.

(i) Since \( f \in B_1(\gamma) \), we can write from (1.2)

\[
zf'(z^2) = f(z^2) \sqrt{\gamma} h(z^2)
\]

where \( \text{Re } h(z) > 0 \), for \( z \in D \).

Set

\[
h(z^2) = \frac{1 + \omega(z^2)}{1 - \omega(z^2)},
\]

where \( \omega(z^2) \) is regular, \( |\omega(z^2)| < 1 \) in \( D \), \( \omega(0) = 0 \) and \( \omega(z) = \sum_{n=1}^{\infty} \omega_n z^n \).

Then with \( F(z) = f(z^2) \sqrt{\gamma} \)

\[
F(z) = z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad (2.1)
\]

gives

\[
(zf'(z^2) + F(z)) \omega(z^2) = zf'(z^2) - F(z).
\]

Thus

\[
\{2z + \frac{\omega}{k=2} \left( \frac{k_a_k + b_{2k-1}^2}{k^2} z^{2k-1} \right) \omega(z^2) = \frac{\omega}{k=2} \left( \frac{k_a_k - b_{2k-1}^2}{k^2} z^{2k-1} \right) \}
\]

Equating coefficients of \( z^{2n-1} \) in (2.2), we find that for \( n \geq 2 \)

\[
n_{n-1} a_n - b_{2n-1} = 2 \omega_{n-1} + (2a_2 + b_3) \omega_{n-2} + \ldots + [(n-4)a_{n-4} + b_{2n-6}] \omega_1.
\]

This means that the coefficient combination \( n_{a_n} - b_{2n-1} \) on the left hand side of (2.2) depends only on the coefficient combinations \( [2a_2 + b_3], \ldots, [(n-4)a_{n-4} + b_{2n-6}] \) on the right-hand side. Hence, for \( n \geq 2 \) we can write

\[
\{2z + \frac{n_{n-1}}{k=2} \left( \frac{k_a_k + b_{2k-1}^2}{k^2} z^{2k-1} \right) \omega(z^2) = \frac{n}{k=2} \left( \frac{k_a_k - b_{2k-1}^2}{k^2} z^{2k-1} \right) \}
\]
say. Squaring the moduli of both sides of (2.3) and integrating round \( |z| = r \), we
obtain, using the fact that \( |w(z^2)| < 1 \) for \( z \in D \),
\[
\sum_{k \geq 2} \left| a_k - b_k \right|^2 \leq 4 \left( \frac{n-1}{k} \right)^2 \sum_{k \geq 2} \left| a_k \right| \left| b_k \right|
\]
and
\[
\sum_{k \geq 2} \left| a_k \right|^2 \leq 4 \left( \frac{n-1}{k} \right)^2 \sum_{k \geq 2} \left| a_k \right|^2
\]
for \( 0 < r < 1 \), and so
\[
\left| a_n - b_{2n-1} \right|^2 \leq 4 \left( \frac{n-1}{k} \right)^2 \left( \frac{1}{k} \right)^4 \left( \frac{n}{k} \right)^2 \left( \frac{2}{2k-1} \right)^2 r^{4k-2}
\]
for \( 0 < r < 1 \), and so
\[
\left| a_n - b_{2n-1} \right|^2 \leq 4 r^{4n+1} \sqrt{\frac{A(r^2, f)}{\pi}} \sqrt{\frac{A(r, F)}{\pi}}
\]
for \( 0 < r < R \).

Lemma 1 now gives
\[
\left| a_n - b_{2n-1} \right|^2 \leq 4 r^{4n+1} \sqrt{A(r^2, f)} \sqrt{A(r, F)}
\]
and choosing \( r = 1 - \frac{1}{n} \) we obtain
\[
\left| a_n - b_{2n-1} \right|^2 \leq C A(1 - \frac{1}{n}) \text{, where } C \text{ is constant.}
\]
Finally, it is easy to see that from the definition of \( B_1(1) \), \( F \in R \), and so [8] for
\( n \leq 2 \), \( \left| b_{2n-1} \right| \leq \frac{2}{2n-1} \). Thus (2.5) gives
\[
n \left| a_n \right| = O(1) + O(1) \sqrt{A(1 - \frac{1}{n})} \text{ as } n \to \infty.
\]
This proves part (i) of Theorem 1.

(ii)

Since \( L(r) = \int_0^{2\pi} \left| z f'(z) \right| d\theta \), and \( F(z)^2 = f(z^2) \), (2.1) gives
\[
L(r^2) = \int_0^{2\pi} \left| z^2 f'(z^2) \right| d\theta \leq r \int_0^{2\pi} \left| F(z) h(z^2) \right| d\theta
\]
and
\[
= I_1(r) + I_2(r) \text{ say.}
\]
Again using (2.1) we have
\[
I_1(r) = r \int_0^{2\pi} \left| h(z^2) \right|^2 d\theta \leq 2\pi r \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \left| h_n \right|^2 r^n \right) d\theta
\]
where \( h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \) for \( z \in D \), and since \( \left| h_n \right| \leq 2 \) for \( n \geq 1 \) [2 p. 10],
\[
I_1(r) \leq 2\pi r \int_0^r (1 + 4 \sum_{n=1}^\infty \rho^{4n}) \, dp
\]

\[
= 0(1) \log\left(\frac{1}{1-r}\right) \text{ as } r \to 1.
\]

Also

\[
I_2(r) \leq 2r \left( \int_0^r \int_0^{2\pi} |F(z)|^2 |h'(z^2)| \, \rho \, d\theta \, dp \right)^{\frac{1}{2}} \left( \int_0^r \int_0^{2\pi} |h'(z^2)| \, \rho \, d\theta \, dp \right)^{\frac{1}{2}}
\]

\[
= 2r \left( J_1(r) \right)^{\frac{1}{2}} \left( J_2(r) \right)^{\frac{1}{2}} \text{ say.}
\]

Since Re \( h(z^2) > 0 \), for \( z \in D \), we may write

\[
h(z^2) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + z^2 e^{-it}}{1 - z^2 e^{-it}} \, du(t),
\]

where \( u(t) \) increases and \( \frac{1}{2\pi} \int_0^{2\pi} \, du(t) = 1. \) [5 p. 68].

Therefore

\[
h'(z^2) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-it}}{(1 - z^2 e^{-it})^2} \, du(t),
\]

and so

\[
|h'(z^2)| \leq \frac{1}{\pi} \int_0^{2\pi} \frac{\, du(t)}{|1 - z^2 e^{-it}|^2}.
\]

Thus

\[
J_1(r) \leq \frac{1}{\pi} \int_0^r \int_0^{2\pi} \left| \frac{F(z)}{1 - z^2 e^{-it}} \right|^2 \, \rho \, d\theta \, dp \cdot du(t).
\]

Since \( F \) is an odd function we may write

\[
\frac{F(z)}{1 - z^2 e^{-it}} = \sum_{n=1}^{\infty} n z^{2n-1}
\]

and so

\[
\int_0^{2\pi} \int_0^{2\pi} \left| \frac{F(z)}{1 - z^2 e^{-it}} \right|^2 \, d\theta \, dp = 2\pi \sum_{n=1}^{\infty} \rho^{4n-2} \int_0^r \int_0^{2\pi} |S_{2n-1}(t)|^2 \, \rho \, d\theta \, dp \, du(t).
\]

We now show that for \( n \geq 1, \)

\[
\int_0^{2\pi} |S_{2n-1}(t)|^2 \, du(t) \leq 2\pi \sum_{j=1}^{n} |a_j| \cdot |b_{2j-1}|,
\]

where \( F(z) = \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1} \) for \( z \in D. \)

From (2.7) we have

\[
\sum_{n=1}^{\infty} S_{2n-1}(t) \cdot z^{2n-1} = \left( \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1} \right) \cdot \left( \sum_{n=0}^{\infty} e^{-int} z^n \right)
\]

and so for \( n \geq 1, \)

\[
S_{2n-1}(t) = \sum_{k=1}^{K} b_{2k-1} e^{-i(n-k)t}.
\]

Now (2.1) gives

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{n} a_n \cdot \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1} = \left( \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1} \right) \left( \sum_{n=0}^{\infty} \rho^{4n-2} \right)
\]

(where \( h_0 = 1 \)) and so for \( n \geq 1, \)

\[
na_n = \sum_{n=1}^{N} b_{2n-1} h_n.
\]

It is easy to see from (2.6) that, for \( k \geq 1, \)
\[ h_k = \frac{1}{\pi} \int_0^{2\pi} e^{-ikt} \, d\nu(t). \]

and so

\[ n a_n = \frac{1}{\pi} \sum_{v=1}^n b_{2v-1} \int_0^{2\pi} e^{-i(n-v)t} \, dt. \]  

(2.9)

Now

\[ |S_{2n}(t)|^2 = S_{2n-1}(t) \overline{S_{2n-1}(t)} \]

\[ = 2 \text{Re} \left[ \sum_{j=1}^n \frac{1}{k^{j-1}} b_{2j-1} \overline{b}_{2k-1} e^{-i(k-j)t} \right] - \sum_{v=1}^n |b_{2v-1}|^2 \]

where we have used (2.8). Therefore, using (2.9)

\[ \int_0^{2\pi} |S_{2n-1}(t)|^2 \, d\nu(t) = \int_0^{2\pi} \left[ 2 \text{Re} \left[ \sum_{j=1}^n \frac{1}{k^{j-1}} b_{2j-1} \overline{b}_{2k-1} e^{-i(k-j)t} \right] - \sum_{v=1}^n |b_{2v-1}|^2 \right] \, d\nu(t) \]

\[ \leq 2 \int_0^{2\pi} \left[ \sum_{j=1}^n |a_j|^2 \right] \, d\nu(t) \]

Thus

\[ J_1(r) \leq \frac{4}{\pi-2} \int_0^r \left[ \sum_{n=1}^{4n-1} \left( \sum_{j=1}^n |a_j|^2 \right) \right] \, dp \]

\[ \leq \frac{4}{\pi-2} \int_0^r \left[ \sum_{n=1}^{4n-1} \left( \sum_{j=1}^n |a_j|^2 \right) \right] \, dp \]

\[ \leq \frac{4}{\pi-2} \int_0^r \left[ \sum_{n=1}^{4n-1} \left( \sum_{j=1}^n |a_j|^2 \right) \right] \, dp \]

\[ \leq \frac{4}{\pi-2} \int_0^r \left[ \sum_{n=1}^{4n-1} \left( \sum_{j=1}^n |a_j|^2 \right) \right] \, dp \]

Since \( A(\rho) \) is increasing on \((0,1)\)

\[ J_1(r) \leq \frac{4}{\pi-2} A(r,f) \log \left( \frac{1-r}{1-\rho} \right). \]

Now

\[ J_2(r) = \int_0^r \int_0^{2\pi} \left| h'(z^2) \right| \rho \, d\theta \, dp \]

and since

\[ \left| h'(z^2) \right| \leq \frac{\text{Re} \, h(z^2)}{1-r^a} \]  

for \(0 < r < 1\),

\[ J_2(r) \leq 2 \int_0^r \int_0^{2\pi} \frac{\text{Re} \, h(z^2)}{1-r^a} \rho \, d\theta \, dp \]

\[ \leq 4 \pi \int_0^r \frac{dp}{1-r^a} \]

since \( h \) is harmonic in \( D \).

Thus

\[ J_2(r) \leq 4 \pi \log \frac{1}{1-r}. \]
Combining the estimates for \( J_1(r) \) and \( J_2(r) \) shows that

\[
I_2(r) = O(1) \sqrt{A(r)} \log \left( \frac{1}{1-r} \right) \text{ as } r \to 1
\]

and the result is proved.

**COROLLARY 1.** Let \( f \in B_1(\frac{1}{2}) \), then as \( n \to \infty \),

(i) \( n|a_n| \leq o(1) + O(1) \frac{1}{n} \),

where for \( 0 \leq r < 1 \), \( P(r) = \sum_{n=1}^{\infty} \frac{|a_n|r^n}{n} \).

(ii) \( n|a_n| \leq o(1) + O(1) A(1 - \frac{1}{n})^{\frac{1}{2}} (\log n)^{\frac{1}{2}} \),

(iii) \( L(r) = O(1) A(r)^{\frac{1}{2}} (\log(\frac{1}{1-r}))^{\frac{8}{4}} \text{ as } r \to 1 \).

**PROOF.** From (2.4) and the fact that \( |b_{2k-1}| \leq \frac{2}{2k-1} \) for \( k \geq 1 \), it follows that, for \( 0 < r < 1 \),

\[
|a_n - b_{2n-1}|^2 \leq 8 \sum_{k=1}^{n} \frac{|a_{2k-1}|^2}{2k-1} \leq 4r^{-n} P(r).
\]

Choosing \( r = 1 - \frac{1}{n} \) (i) follows.

(ii) follows since

\[
P(r) \leq \left( \sum_{n=1}^{\infty} \frac{|a_n|^2 r^n}{n} \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \frac{n}{r^n} \right)^{\frac{1}{2}}.
\]

It follows trivially from (2.1) that

\[
L(r) = O(1) M(r)^{\frac{1}{2}} \log \frac{1}{1-r} \text{ as } r \to 1
\]

and so (iii) follows at once on noting that

\[
M(r)^2 \leq \frac{A(\chi)}{\pi} \log \frac{1}{1-r}
\]

and on using lemma 2.

**REMARK.** In view of Theorem 1 and Corollary 1, it is possible that for \( f \in B_1(\frac{1}{2}) \) the following conjectures are valid.

(i) \( n^2 a_n^2 = O(1) M(1 - \frac{1}{n}) \) as \( n \to \infty \),

(ii) \( n^4 a_n^4 = O(1) A(1 - \frac{1}{n}) \) as \( n \to \infty \),

(iii) \( L(r) = O(1) A(r)^{\frac{1}{2}} (\log(\frac{1}{1-r}))^{\frac{8}{4}} \text{ as } r \to 1 \).

We note that (ii) is stronger than (i) and that we have proved (ii) and (iii) in the case when \( A(r) \) is finite.

The following extensions to Theorem 1 support the above conjectures.

3. INTEGRAL MEANS.

For \( f \) regular in \( D \), define for \( \lambda \) real,

\[
l_\lambda(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta.
\]
THEOREM 2. For \( f \in B_1(\frac{1}{2}) \) and \( \lambda > 1 \),

\[
I_\lambda(r^2, zf') \leq C(\lambda) \int_0^r \frac{M(\rho)^{\lambda/2}}{(1-\rho)^{\lambda}} \, d\rho,
\]

where \( C(\lambda) \) is a constant depending only on \( \lambda \).

PROOF. (2.1) gives

\[
I_\lambda(r^2, zf') = \int_0^{2\pi} |z^2 f'(z^2)|^\lambda \, d\theta
\]

\[
\leq \lambda r \int_0^{2\pi} |zf'(z^2)|^{\lambda-1} |F(z) \, h(z^2)| \, d\theta \, d\rho
\]

\[
+ 2\lambda r \int_0^{2\pi} |zf'(z^2)|^{\lambda-1} |F(z) \, h(z^2)| \rho \, d\theta \, d\rho
\]

\[
= J_1^\lambda(r) + J_2^\lambda(r) \text{ by say.}
\]

Now for \( \lambda \geq 1 \),

\[
J_k^\lambda(r) \leq \lambda r \int_0^{2\pi} (M(\rho^2, zf'))^{\lambda-1} \, d\rho \int_0^{2\pi} |F'(z) \, h(z^2)| \, d\theta \, d\rho.
\]

From the proof of Theorem 1 (ii), we have with \( z = \rho e^{i\theta} \),

\[
\int_0^{2\pi} |F'(z) \, h(z^2)| \, d\theta \leq O(1) \frac{1}{1-\rho} \text{ as } \rho \to 1.
\]

Also (2.1) and the distortion theorem for functions of positive real part [4] gives

\[
M(\rho^2, zf') \leq \frac{2\rho M(\rho, F)}{1-\rho}.
\]

Thus

\[
J_1^\lambda(r) \leq C(\lambda) \int_0^{2\pi} \frac{M(\rho, F)^{-1}}{(1-\rho)^{\lambda}} \, d\rho
\]

and since \( F(z)^2 = f(z^2) \),

\[
J_1^\lambda(r) \leq C(\lambda) \int_0^{2\pi} \frac{M(\rho, f)^{-1}}{(1-\rho)^{\lambda}} \, d\rho
\]

Similarly, using the fact that \( h \) is harmonic, we have

\[
J_2^\lambda(r) \leq C(\lambda) \int_0^{2\pi} \frac{M(\rho, f)^{\lambda}}{(1-\rho)^{\lambda}} \rho \, d\rho
\]

Combined with the results for \( J_1^\lambda(r) \) and \( J_2^\lambda(r) \) we obtain the result.

THEOREM 3. Let \( f \in B_1(\frac{1}{2}) \), then for \( 0 \leq r \leq 1 \),

\[
I_1(r^2, f) \leq I_1(r^2, f_0)
\]

where

\[
f_0(z^2) = \left( \int_0^{1+rt^2} \frac{1}{1-t^2} \, dt \right)^2
\]

PROOF. Since \( f \in B_1(\frac{1}{2}) \), then \( F \in R \).

Thus

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(z^2)| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |F(z)|^2 \, d\theta
\]

\[
= r^2 + \sum_{n=1}^{\infty} |b_{2n-1}|^2 r^{4n-2}
\]
and since for \( n \geq 1 \), \(|b_{2n-1}| \leq \frac{2}{2n-1}\)

\[
I_1(r^2, f) \leq r^2 + 4 \sum_{n=2}^{\infty} \frac{r^{4n-2}}{(2n-1)^2} = \frac{1}{2\pi} \int_0^\pi |f_0(z^2)| d\theta
\]

and the theorem is proved.

REFERENCES

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