ON PAIRWISE S-CLOSED BITOPOLOGICAL SPACES

M. N. MUKHERJEE

Department of Mathematics
Charu Chandra College
22 Lake Road
Calcutta, India 700 029

(Received August 5, 1982)

ABSTRACT. The concept of pairwise S-closedness in bitopological spaces has been introduced and some properties of such spaces have been studied in this paper.

KEY WORDS AND PHRASES. Pairwise semi-open, Pairwise almost compact, Pairwise S-closed, Pairwise regularly open and regularly closed, Pairwise extremally disconnectedness, Pairwise semi-continuous and irresolute functions.

1980 AMS MATHEMATICS SUBJECT CLASSIFICATION CODES. 54E55.

1. INTRODUCTION.

Travis Thompson [1] in 1976 initiated the notion of S-closed topological spaces, which was followed by its further study by Thompson [2], T. Noiri [3,4] and others. It is now the purpose of this paper to introduce and investigate the corresponding concept, i.e., pairwise S-closedness in bitopological spaces. To make the exposition of this paper self-contained as far as possible, we shall quote some definitions and enunciate some theorems from [5,6,7].

DEFINITION 1.1. [7] Let \((X, \tau_1, \tau_2)\) be a bitopological space.

(i) A subset \(A\) of \(X\) is called \(\tau_i\) semi-open with respect to \(\tau_j\) (abbreviated as \(s.o.w.r.t. \tau_j\)) in \(X\) if there exists a \(\tau_i\) open set \(B\) such that \(B \subset A \subset \overline{B}^{\tau_j}\) (where \(\overline{B}^{\tau_j}\) denotes the \(\tau_j\)-closure of \(B\) in \(X\)), where \(i, j = 1, 2\) and \(i \neq j\).

\(A\) is called pairwise semi-open (written as \(p.s.o\)) in \(X\) if \(A\) is \(\tau_1\) s.o.w.r.t. \(\tau_1\) as well as \(\tau_2\) s.o.w.r.t. \(\tau_1\) in \(X\).
(ii) A subset $A$ of $X$ is called $\tau_1$ semi-closed with respect to $\tau_2$ (denoted as $\tau_1$ s.c.l.w.r.t. $\tau_2$) if $X - A$ is $\tau_1$ s.o.w.r.t. $\tau_2$. Definitions for $\tau_2$ s.c.l.w.r.t. $\tau_1$ and p. s.c.l. sets can be given similarly as in (i).

(iii) A subset $N$ of $X$ is called a $\tau_i$ semi-neighborhood of $x$ w.r.t. $\tau_j$, where $x \in X$, if there is a $\tau_i$ s.o. set w.r.t. $\tau_j$ containing $x$ and contained in $N$. A point $x$ of $X$ is said to be a $\tau_i$ semi-accumulation point of a subset $A$ of $X$ w.r.t. $\tau_j$, if every $\tau_i$ semi-neighborhood of $x$ w.r.t. $\tau_j$ intersects $A$ in at least one point other than $x$, where $i, j = 1, 2$ and $i \neq j$.

(iv) The intersection of all $\tau_i$ s.c.l. sets w.r.t. $\tau_j$, each containing a subset $A$ of $X$, is called the $\tau_i$ semi-closure of $A$ w.r.t. $\tau_j$ and will be denoted by $A_{\tau_i}(\tau_j)$, where $i, j = 1, 2$ and $i \neq j$.

It has been proved in [7] that a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_i$ s.c.l.w.r.t. $\tau_j$ if and only if $A = A_{\tau_i}(\tau_j)$ and moreover, $x \in A_{\tau_i}(\tau_j)$ if and only if $x$ is either a point of $A$ or a $\tau_i$ semi-accumulation point of $A$ w.r.t. $\tau_j$, where $i \neq j$ and $i, j = 1, 2$.

In [7], it was deduced that $A \subseteq (X, \tau_1, \tau_2)$ is $\tau_1$ s.o.w.r.t. $\tau_2$ iff $\overline{A^{i2}} = (A^1)^{i2}$, where $A^1$ denotes the $\tau_1$-interior of $A$ in $X$. Similarly we shall use $A^{i2}$ to mean the $\tau_2$-interior of $A$ in $X$.

It is very easy to see that every $\tau_i$ open set in $(X, \tau_1, \tau_2)$ is $\tau_i$ s.o.w.r.t. $\tau_j$ and the union of any collection of sets that are $\tau_i$ s.o.w.r.t. $\tau_j$, is also so, where $i, j = 1, 2$; $i \neq j$. It was shown in [5] that the intersection of two $\tau_1$ s.o. sets w.r.t. $\tau_2$ is not necessarily $\tau_1$ s.o.w.r.t. $\tau_2$. But we have,

THEOREM 1.2. [5] If $A$ is $\tau_i$ s.o.w.r.t. $\tau_j$ in $(X, \tau_1, \tau_2)$ and $B \subseteq T_1 \cap T_2$, then $A \cap B$ is $\tau_i$ s.o.w.r.t. $\tau_j$, where $i, j = 1, 2$ and $i \neq j$.

The first part of the following theorem was proved in [7] and the converse part in [5].

THEOREM 1.3. Let $A \subseteq Y \subseteq (X, \tau_1, \tau_2)$. If $A$ is $\tau_i$ s.o.w.r.t. $\tau_j$, then $A$ is $(\tau_i)_Y$ s.o.w.r.t. $(\tau_j)_Y$. Conversely, if $A$ is $(\tau_i)_Y$ s.o.w.r.t. $(\tau_j)_Y$ and $Y \subseteq T_1$, then $A$ is $\tau_i$ s.o.w.r.t. $\tau_j$, where $i, j = 1, 2$ and $i \neq j$.

DEFINITION 1.4. [6] (a) A bitopological space $(X, \tau_1, \tau_2)$ is said to be $\tau_i$ almost compact w.r.t. $\tau_j$ ($i, j = 1, 2$; $i \neq j$) if every $\tau_i$ open filterbase has a $\tau_j$ cluster point. $(X, \tau_1, \tau_2)$ is called pairwise almost compact if it is $\tau_i$
almost compact w.r.t. $\tau_2$ and $\tau_2$ almost compact w.r.t. $\tau_1$.

(b) A bitopological space $(X^*, \tau^*_1, \tau^*_2)$ is called an extension of a bitopological space $(X, \tau_1, \tau_2)$ if $X \subseteq X^*$, $\tau^*_i = X^*$ and $(\tau^*_i)_X = \tau_i$, for $i = 1, 2$.

A pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is called pairwise H-closed if the space cannot have any pairwise Hausdorff extension.

**Theorem 1.5.** [6] (a) $(X, \tau_1, \tau_2)$ is pairwise almost compact if and only if for each cover $\{G_a : a \in A\}$ of $X$ by $\tau_i$ open sets, there exists a finite subcollection $\{G_{a_1}, \ldots, G_{a_n}\}$ such that $X = \bigcup_{k=1}^n G_{a_k}$, where $i, j = 1, 2$ and $i \neq j$.

(b) If $(X, \tau_1, \tau_2)$ is $\tau_i$ regular w.r.t. $\tau_j$ and $\tau_i$ almost compact w.r.t. $\tau_j$, then $(X, \tau_i)$ is compact, for $i, j = 1, 2$ and $i \neq j$.

(c) A pairwise Hausdorff and pairwise almost compact bitopological space is pairwise H-closed.

In what follows, by $(X, \tau_1, \tau_2)$ we shall always mean a bitopological space, i.e., a set $X$ endowed with two topologies $\tau_1$ and $\tau_2$.

2. PAIRWISE S-CLOSED SPACES.

**Definition 2.1.** Let $F = \{F_a\}$ be a filterbase in $(X, \tau_1, \tau_2)$ and $x \in X$. $F$ is said to

(i) $\tau_i$ S-accumulate to $x$ w.r.t. $\tau_j$ if for every $\tau_i$ s.o. set $V$ w.r.t. $\tau_j$ containing $x$ and each $F_a \in F$, $F_a \cap V^{\tau_j} \neq \emptyset$.

(ii) $\tau_i$ S-converge w.r.t. $\tau_j$ to $x$, if corresponding to each $\tau_i$ s.o. set $V$ w.r.t. $\tau_j$ containing $x$, there exists $F_a \in F$ such that $F_a \subseteq V^{\tau_j}$.

In (i) and (ii) above, $i \neq j$ and $i, j = 1, 2$. $F$ is said to pairwise S-converge to $x$ if $F$ is $\tau_1$ S-convergent to $x$ w.r.t. $\tau_2$ as well as $\tau_2$ S-convergent to $x$ w.r.t. $\tau_1$. The definition of pairwise S-accumulation point of $F$ is similar.

**Definition 2.2.** $(X, \tau_1, \tau_2)$ is called $\tau_1$ S-closed w.r.t. $\tau_2$ if for each cover $\{V_a : a \in A\}$ of $X$ with $\tau_1$ s.o. sets w.r.t. $\tau_2$, there is a finite subfamily $\{V_{a_i} : i = 1, 2, \ldots, n\}$ such that $\bigcup_{i=1}^n V_{a_i} = X$ (where $I$ is some index set). $X$ is called pairwise S-closed if it is $\tau_1$ S-closed w.r.t. $\tau_2$ and $\tau_2$ S-closed w.r.t. $\tau_1$.

**Theorem 2.3.** Let $F$ be an ultrafilter in $X$. Then $F$ $\tau_1$ S-accumulates to a point
$x_0 \in X$ w.r.t. $\tau_2$ if and only if $F$ is $\tau_1$-convergent to $x_0$ w.r.t. $\tau_2$.

**Proof:** Let $F$ be $\tau_1$-convergent w.r.t. $\tau_2$ to $x_0$ and let it not $\tau_1$-S-accumulate w.r.t. $\tau_2$ to $x_0$. Then there exist a $\tau_1$-s.o. set $V$ w.r.t. $\tau_2$ (containing $x_0$) and some $F_a \in F$ such that $F_a \cap \overset{\tau_2}{V} = \emptyset$. Then $F_a \subseteq X - \overset{\tau_2}{V}$ and hence

$$X - \overset{\tau_2}{V} \in F. \quad (2.1)$$

Since $F$ is $\tau_1$-convergent w.r.t. $\tau_2$ to $x_0$, corresponding to $V$ there exists $F_\beta \in F$ such that $F_\beta \subseteq \overset{\tau_2}{V}$. Then $\overset{\tau_2}{V} \in F. \quad (2.2)$ Clearly (2.1) and (2.2) are incompatible. Note that for this part we do not need maximality of $F$.

Conversely, if $F$ does not $\tau_1$-converge w.r.t. $\tau_2$ to $x_0$, there exists a $\tau_1$-s.o. set $V$ w.r.t. $\tau_2$ containing $x_0$, such that $F_a \notin \overset{\tau_2}{V}$, for each $F_a \in F$. But $F$ has $x_0$ as a $\tau_1$-accumulation point w.r.t. $\tau_2$. Hence $F_a \cap \overset{\tau_2}{V} \neq \emptyset$, for each $F_a \in F$. Thus $F_a \cap \overset{\tau_2}{V} \neq \emptyset$ and $F_a \cap (X - \overset{\tau_2}{V}) \neq \emptyset$, for each $F_a \in F$. Since $F$ is maximal, this shows that $\overset{\tau_2}{V}$ and $X - \overset{\tau_2}{V}$ both belong to $F$, which is a contradiction.

**Note 2.4.** In the above theorem, the indices 1 and 2 could be interchanged.

**Theorem 2.5.** In a bitopological space $(X, \tau_1, \tau_2)$ the following are equivalent:

(a) $X$ is $\tau_1$-S-closed w.r.t. $\tau_2$.

(b) Every ultrafilterbase $F$ is $\tau_1$-convergent w.r.t. $\tau_2$.

(c) Every filterbase $\tau_1$-accumulates w.r.t. $\tau_2$ to some point of $X$.

(d) For every family $(F_a)$ of $\tau_1$-s.o. sets w.r.t. $\tau_2$, with $\bigcap F_a = \emptyset$, there exists a finite subcollection $(F_{a_i})^n_{i=1}$ of $(F_a)$ such that $\bigcap_{i=1}^n (F_{a_i})^{\tau_2} = \emptyset$.

**Proof:** (a) $\Rightarrow$ (b) Let $F = (F_a)$ be an ultrafilterbase in $X$, which does not $\tau_1$-converge w.r.t. $\tau_2$ to any point of $X$. Then by Theorem 2.3, $F$ has no $\tau_1$-S-accumulation point w.r.t. $\tau_2$. Thus for every $x \in X$, there is a $\tau_1$-s.o. set $V(x)$ w.r.t. $\tau_2$ containing $x$ and an $F_{a(x)} \in F$ such that $F_{a(x)} \cap \overset{\tau_2}{V(x)} = \emptyset$.

Evidently, $(V(x): x \in X)$ is a cover of $X$ with sets that are $\tau_1$-s.o.w.r.t. $\tau_2$ and by (a), there exists a finite subcollection $(V(x_i): i = 1, 2, \ldots, n)$ of $(V(x): x \in X)$ such that $\overset{n}{\bigcup}_{i=1} V(x_i)^{\tau_2} = X$.

Now, $F$ being a filterbase, there exists $F_0 \in F$ such that

$$F_0 \subseteq \overset{n}{\bigcap}_{i=1} F_{a(x_i)}.$$
PAIRWISE S-CLOSED BITOPOLOGICAL SPACES
733

Then \( F_0 \cap V(x_i)^{\tau_2} = \emptyset \) for \( i = 1, 2, \ldots, n \).

\[
\Rightarrow F_0 \cap \left( \bigcup_{i=1}^{n} V(x_i)^{\tau_2} \right) = F_0 \cap X = \emptyset \Rightarrow F_0 = \emptyset \] which is a contradiction.

(b) \( \Rightarrow \) (c) Every filterbase \( F \) is contained in an ultrafilter base \( F^* \) and \( F^* \) is \( \tau_1 \) S-convergent w.r.t. \( \tau_2 \) to some point \( x_0 \) by (b), and hence \( x_0 \) is a \( \tau_1 \) S-accumulation point of \( F^* \) w.r.t. \( \tau_2 \). Since \( F \subset F^* \), \( x_0 \) is also a \( \tau_1 \) S-accumulation point of \( F \) w.r.t. \( \tau_2 \).

(c) \( \Rightarrow \) (d) Let \( F = (F_a) \) be a family of \( \tau_1 \) s.c.l. sets w.r.t. \( \tau_2 \) with \( \bigcap F_a = \emptyset \) and be such that for every finite subfamily \( \{F_{a_i}\}_{i=1}^{n} \) (say), \( \bigcap_{i=1}^{n} F_{a_i}^{\tau_2} \neq \emptyset \). Thus \( F = \bigcap_{i=1}^{n} F_{a_i}^{\tau_2} \). \( n = \) positive integer, \( F_{a_i} \in F \) forms a filterbase in \( X \) and hence by hypothesis has a \( \tau_1 \) S-accumulation point \( x_0 \) w.r.t. \( \tau_2 \). Then for any \( \tau_1 \) s.o. set \( V(x_0) \) w.r.t. \( \tau_2 \) containing \( x_0 \), \( (F_{a_0})^{\tau_2} \bigcap V(x_0)^{\tau_2} \neq \emptyset \), for each \( F_{a_0} \in F \). Since \( \bigcap F_a = \emptyset \), there is some \( F_{a_0} \in F \) such that \( x_0 \notin F_{a_0} \). Hence \( x_0 \in X - F_{a_0} \) which is \( \tau_1 \) s.o.w.r.t. \( \tau_2 \). Hence \( (F_{a_0})^{\tau_2} \bigcap (X - F_{a_0})^{\tau_2} \neq \emptyset \) or,

\[
(F_{a_0})^{\tau_2} \bigcap (X - (F_{a_0})^{\tau_2}) \neq \emptyset \] which is impossible.

(d) \( \Rightarrow \) (a) Let \( \{V_a\} \) be a covering of \( X \) with sets that are \( \tau_1 \) s.o.w.r.t. \( \tau_2 \). Then \( \bigcap (X - V_a) = X - \bigcup V_a = \emptyset \). By (d), there exists finite number of indices \( a_1, a_2, \ldots, a_n \) such that \( \bigcap_{k=1}^{n} (X - V_{a_k})^{\tau_2} = \emptyset \), i.e., \( \bigcap_{k=1}^{n} (X - V_{a_k})^{\tau_2} = \emptyset \) or,

\[
X - \bigcup_{k=1}^{n} V_{a_k}^{\tau_2} = \emptyset, \text{ or } \bigcup_{k=1}^{n} V_{a_k}^{\tau_2} = X \] and hence \( X \) is \( \tau_1 \) S-closed w.r.t. \( \tau_2 \).

NOTE 2.6. Obviously, in the above theorem, the indices 1 and 2 could have been interchanged and hence the statement (a) can be replaced by "\( X \) is pairwise S-closed" with corresponding alterations in (b), (c) and (d).

DEFINITION 2.7. A subset \( Y \) of \( (X, \tau_1, \tau_2) \) will be called \( \tau_1 \) S-closed w.r.t. \( \tau_j \) in \( X \) if and only if for every cover \( \{V_a: a \in I\} \) of \( Y \) by \( \tau_1 \) s.o. sets w.r.t. \( \tau_j \) of \( X \), there exists a finite set of indices \( a_1, a_2, \ldots, a_n \in I \) such that \( Y \subset \bigcup_{k=1}^{n} V_{a_k}^{\tau_j} \), where \( i, j = 1, 2 \) and \( i \neq j \).
THEOREM 2.8. A subset $Y$ of $(X, \tau_1, \tau_2)$ will be $(\tau_i)_Y$ S-closed w.r.t. $(\tau_j)_Y$ if $Y$ is $\tau_i$ S-closed w.r.t. $\tau_j$ in $X$ and $Y \in \tau_i$, where $i, j = 1, 2$ and $i \neq j$.

PROOF: We prove the theorem by taking $i = 1$ and $j = 2$. Similar will be the proof when $i = 2$ and $j = 1$. By virtue of Theorem 1.3, every cover $\{V_a: a \in I\}$ of $Y$ by sets that are $(\tau_1)_Y$ s.o.w.r.t. $(\tau_2)_Y$ can be regarded as a cover of $Y$ by sets that are $\tau_1$ s.o.w.r.t. $\tau_2$. Then by hypothesis, there is a finite number of indices $a_1, a_2, \ldots, a_n$ such that

$$Y \subseteq \bigcup_{k=1}^{n} \overline{V_{a_k}}^{\tau_2} \Rightarrow Y = \bigcup_{k=1}^{n} \overline{V_{a_k}}^{(\tau_2)_Y}$$

and the theorem follows.

THEOREM 2.9. If $Y \subset (X, \tau_1, \tau_2)$ is $(\tau_i)_Y$ S-closed w.r.t. $(\tau_j)_Y$ and $Y \in \tau_1 \cap \tau_2$, then $Y$ is $\tau_i$ S-closed w.r.t. $\tau_j$ in $X$, for $i, j = 1, 2$ and $i \neq j$.

PROOF: We prove only the case when $i = 1$ and $j = 2$. Let $\{G_a\}$ be a cover of $Y$, where each $G_a$ is $\tau_1$ s.o.w.r.t. $\tau_2$. Then by Theorem 1.2, $G_a \cap Y$ is $\tau_1$ s.o.w.r.t. $\tau_2$ for each $a$ and hence by Theorem 1.3, $G_a \cap Y$ is $(\tau_1)_Y$ s.o.w.r.t. $(\tau_2)_Y$ for each $a$. By hypothesis, there exists a finite number of indices $a_1, a_2, \ldots, a_n$ such that

$$Y = \bigcup_{k=1}^{n} (G_{a_k} \cap Y)^{\tau_2} \Rightarrow Y \subseteq \bigcup_{k=1}^{n} \overline{G_{a_k}}^{\tau_2} \Rightarrow Y \text{ is } \tau_1 \text{ S-closed w.r.t. } \tau_2 \text{ in } X.$$ 

DEFINITION 2.10. [7] A subset $A$ in $(X, \tau_1, \tau_2)$ is called $\tau_1$ regularly open (closed) w.r.t. $\tau_2$ if and only if $A = (A^\tau_2)^{\tau_1}$ (respectively if and only if $A = (A^\tau_1)^{\tau_2}$). Similarly we define sets that are $\tau_2$ regularly open (closed) w.r.t. $\tau_1$.

It has been shown in [7] that a subset $B$ of $(X, \tau_1, \tau_2)$ is $\tau_i$ regularly closed w.r.t. $\tau_j$ iff $(X - B)$ is $\tau_i$ regularly open w.r.t. $\tau_j$, for $i, j = 1, 2$ and $i \neq j$.

LEMMA 2.11. If a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_j$ regularly closed w.r.t. $\tau_i$, then $A$ is $\tau_i$ s.o.w.r.t. $\tau_j$, where $i, j = 1, 2$ and $i \neq j$.

PROOF: Proof is done only in the case when $i = 1$ and $j = 2$.

$A$ is $\tau_2$ regularly closed w.r.t. $\tau_1 \Rightarrow (X - A)$ is $\tau_2$ regularly open w.r.t. $\tau_1$

$$\Rightarrow X - A = \left(\overline{(X - A)}^{\tau_1}\right)^{\tau_2} \quad (2.3)$$
Let $0 = X - \overline{(X - A)^{\tau_1}}$. Then $0$ is $\tau_1$ open and

$$0^{\tau_2} = \overline{\left[ X - \overline{(X - A)^{\tau_1}} \right]^{\tau_2}} = \overline{X - \left[ X - \overline{(X - A)^{\tau_1}} \right]^{\tau_2}} = A \text{ (by (2.3)).}$$

Thus $0 \subset A \subset 0^{\tau_2}$ and $0 \in \tau_1$. Hence $A$ is $\tau_1$ s.o.w.r.t. $\tau_2$.

**LEMMA 2.12.** If a subset $A$ of $(X, \tau_1, \tau_2)$ is $\tau_1$ s.o.w.r.t. $\tau_j$ then $A^{\tau_j}$ is $\tau_j$ regularly closed w.r.t. $\tau_i$, where $i \neq j$ and $i, j = 1, 2$.

**PROOF:** As before we consider the case $i = 1$ and $j = 2$. Since $A$ is $\tau_1$ s.o.w.r.t. $\tau_2$, we have $A^{\tau_1} \subset A \subset A^{\tau_2}$. Then $A^{\tau_2} = (A^{\tau_1})^{\tau_2}$ .... (2.4)

It has been shown in [7] that a set $A$ in $(X, \tau_1, \tau_2)$ is $\tau_1$ regularly closed w.r.t. $\tau_j$ ($i, j = 1, 2; i \neq j$) if it is $\tau_1$ closure of some $\tau_j$ open set. Since $A^{\tau_1}$ is $\tau_1$ open, by virtue of (2.4) the result follows.

**THEOREM 2.13.** A bitopological space $(X, \tau_1, \tau_2)$ is $\tau_1$ S-closed w.r.t. $\tau_j$ if and only if every proper $\tau_j$ regularly open set w.r.t. $\tau_1$ of $X$ is $\tau_1$ S-closed w.r.t. $\tau_j$, for $i, j = 1, 2$ and $i \neq j$.

**PROOF:** We only take up the case $i = 1$ and $j = 2$.

Let $X$ be $\tau_1$ S-closed w.r.t. $\tau_2$ and $F$ be a proper $\tau_2$ regularly open set of $X$ w.r.t. $\tau_1$. Let $\{V_\alpha : \alpha \in I\}$ be a cover of $F$ by sets that are $\tau_1$ s.o.w.r.t. $\tau_2$. Since $X - F$ is $\tau_1$ regularly closed w.r.t. $\tau_1$, by Lemma 2.11, $(X - F)$ is $\tau_1$ s.o.w.r.t. $\tau_2$ and hence $(X - F) \cup \{V_\alpha : \alpha \in I\}$ is a cover of $X$ by $\tau_1$ s.o. sets w.r.t. $\tau_2$. Since $X$ is $\tau_1$ S-closed w.r.t. $\tau_2$, there exists a finite-number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = (X - F)^{\tau_2} \cup \bigcup_{k=1}^{n} (V_{\alpha_k}^{\tau_2})$.

Since $F$ is $\tau_2$ open, $F \cap X - F^{\tau_2} = \emptyset$ and hence $F \subset \bigcup_{k=1}^{n} (V_{\alpha_k}^{\tau_2})$, proving that $F$ is $\tau_1$ S-closed w.r.t. $\tau_2$. Conversely, let $\{V_\alpha : \alpha \in I\}$ be a cover of $X$ by sets that are $\tau_1$ s.o.w.r.t. $\tau_2$. If $X = V_\alpha^{\tau_2}$ for each $\alpha \in I$, then the theorem is proved. So, suppose $X \neq V_\alpha^{\tau_2}$, for some $\beta \in I$ and $V_\beta \neq \emptyset$. Then $V_\beta^{\tau_2}$ is a proper subset of $X$. Since $V_\beta$ is $\tau_1$ s.o.w.r.t. $\tau_2$, by Lemma 2.12, $V_\beta^{\tau_2}$ is $\tau_2$ regularly closed w.r.t. $\tau_1$, so that $X - V_\beta^{\tau_2}$ is proper $\tau_2$ regularly open w.r.t. $\tau_1$ and by hypothesis, it is $\tau_1$ S-closed w.r.t. $\tau_2$. Then there exists a finite
set of indices \( a_1, a_2, \ldots, a_m \) such that \( X - V_{B}^{\tau_2} \subseteq \bigcup_{k=1}^{m} V_{a_k}^{\tau_2} \). Hence

\[
X = V_{B}^{\tau_2} \cup \left( \bigcup_{k=1}^{m} V_{a_k}^{\tau_2} \right)
\]

and \( X \) is \( \tau_1 \)-closed w.r.t. \( \tau_2 \).

**THEOREM 2.14.** A subset \( A \) in \( (X, \tau_1, \tau_2) \) is \( \tau_1 \)-closed w.r.t. \( \tau_2 \) in \( X \) if and only if every cover of \( A \) by sets that are \( \tau_j \)-regularly closed w.r.t. \( \tau_1 \) in \( X \), has a finite subcover, where \( i, j = 1, 2 \) and \( i \neq j \).

**PROOF:** We consider only the case \( i = 1 \) and \( j = 2 \). Let \( A \) be \( \tau_1 \)-closed w.r.t. \( \tau_2 \) in \( X \) and \( \{V_a\} \) be a collection of \( \tau_2 \)-regularly closed sets in \( X \) w.r.t. \( \tau_1 \), which is a cover of \( A \). Then each \( V_a \) is \( \tau_1 \)-s.o.w.r.t. \( \tau_2 \), by Lemma 2.11 and hence there exists a finite set of indices \( a_1, a_2, \ldots, a_n \) such that

\[
A \subseteq V_{a_1}^{\tau_2} \cup \ldots \cup V_{a_n}^{\tau_2} = V_{a_1} \cup \ldots \cup V_{a_n}
\]

(since each \( V_{a_i} \) is \( \tau_2 \)-closed). Conversely, let the given condition hold and \( \{V_a\} \) be a \( \tau_1 \)-s.o. cover of \( A \) w.r.t. \( \tau_2 \). Then \( V_{a_i}^{\tau_2} \) is \( \tau_2 \)-regularly closed w.r.t. \( \tau_1 \) for each \( a_i \), by Lemma 2.12, and \( \{V_{a_i}^{\tau_2}\} \) is a cover of \( A \). Then by hypothesis, there exist a finite number of indices \( a_1, a_2, \ldots, a_n \) such that \( A \subseteq \bigcup_{k=1}^{n} V_{a_k}^{\tau_2} \), showing that \( A \) is \( \tau_1 \)-closed w.r.t. \( \tau_2 \).

**THEOREM 2.15.** If \( A \) and \( B \) are \( \tau_1 \)-closed w.r.t. \( \tau_j \) in \( (X, \tau_1, \tau_2) \), then \( A \cup B \) is also so, where \( i, j = 1, 2 \) and \( i \neq j \).

**PROOF:** Let \( \{V_a\} \) be a cover of \( A \cup B \) by sets that are \( \tau_i \)-s.o.w.r.t. \( \tau_j \) in \( X \). Then it is a cover of \( A \) as well as of \( B \). By hypothesis, there will exist a finite number of indices \( a_{11}, a_{12}, \ldots, a_{1k} \) and \( a_{21}, a_{22}, \ldots, a_{2r} \) such that

\[
A \subseteq \bigcup_{k=1}^{k} V_{a_{1k}}^{\tau_j} \quad \text{and} \quad B \subseteq \bigcup_{k=1}^{r} V_{a_{2k}}^{\tau_j}.
\]

Then \( A \cup B \subseteq \bigcup_{k=1}^{k} V_{a_{1k}}^{\tau_j} \cup \bigcup_{k=1}^{r} V_{a_{2k}}^{\tau_j} \) and hence \( A \cup B \) is \( \tau_1 \)-closed w.r.t. \( \tau_j \).

**THEOREM 2.16.** If \( A \) is \( \tau_1 \)-closed w.r.t. \( \tau_2 \) in \( (X, \tau_1, \tau_2) \) then \( A^{\tau_2} \) is also so.

**PROOF:** Let \( \{V_a\} \) be a cover of \( A^{\tau_2} \) by sets that are \( \tau_1 \)-s.o.w.r.t. \( \tau_2 \), then it is also a cover of \( A \). Thus there exists a finite number of indices \( a_1, \ldots, a_n \) such that \( A \subseteq \bigcup_{i=1}^{n} V_{a_i}^{\tau_2} \Rightarrow A^{\tau_2} \subseteq \bigcup_{i=1}^{n} V_{a_i}^{\tau_2} \) and the result follows.
Theorem 2.9 and Theorem 2.16 we get:

**COROLLARY 2.17.** If \( (X, \tau_1, \tau_2) \) is pairwise open and \((A, (\tau_1)_A, (\tau_2)_A)\) is pairwise \(S\)-closed, then \(A^\tau_i\) is pairwise \(S\)-closed in \(X\), for \(i = 1, 2\).

**COROLLARY 2.18.** A space \((X, \tau_1, \tau_2)\) is \(\tau_i\) \(S\)-closed w.r.t. \(\tau_j\) if there exists a \(\tau_i\) \(S\)-closed subset \(A\) w.r.t. \(\tau_j\) in \(X\), which is \(\tau_j\) dense in \(X\), where \(i, j = 1, 2\) and \(i \neq j\).

**THEOREM 2.19.** Let \(A \subseteq (X, \tau_1, \tau_2)\) be \(\tau_1\) \(S\)-closed w.r.t. \(\tau_2\) and \(B\) is \(\tau_2\) regularly open w.r.t. \(\tau_1\) in \(X\). Then \(A \cap B\) is \(\tau_1\) \(S\)-closed w.r.t. \(\tau_2\).

**PROOF.** Let \(\{V_{\alpha}: \alpha \in I\}\) be a \(\tau_1\) s.o. cover of \(A \cap B\) w.r.t. \(\tau_2\), where \(I\) is some index set. Since \(X - B\) is \(\tau_2\) regularly closed w.r.t. \(\tau_1\), by Lemma 2.11, \((X - B)\) is \(\tau_1\) s.o.w.r.t. \(\tau_2\). Thus \(A \subseteq \bigcup_{\alpha \in I} (V_{\alpha}) \cup (X - B)\) and \(A\) is \(\tau_1\) \(S\)-closed w.r.t. \(\tau_2\).

Then there exist indices \(a_1, a_2, \ldots, a_n\), finite in number, such that

\[
A \subseteq \bigcup_{i=1}^{n} V_{a_i}^\tau_2 \cup (X - B)^\tau_2 = \bigcup_{i=1}^{n} V_{a_i}^\tau_2 \cup (X - B).
\]

Thus \(A \cap B \subseteq \bigcup_{i=1}^{n} V_{a_i}^\tau_2\) and \(A \cap B\) is \(\tau_1\) \(S\)-closed w.r.t. \(\tau_2\).

**COROLLARY 2.20.** Let \(A \subseteq (X, \tau_1, \tau_2)\) be \(\tau_1\) \(S\)-closed w.r.t. \(\tau_2\) and \(B\) is \(\tau_2\) regularly open w.r.t. \(\tau_1\), then

(a) \(B\) is \(\tau_1\) \(S\)-closed w.r.t. \(\tau_2\) if \(B \subseteq A\).

(b) \(A^\tau_1\) is \(\tau_2\) \(S\)-closed w.r.t. \(\tau_2\) if \(A\) is \(\tau_1\) closed in \(X\).

**PROOF:** (a) Follows immediately from Theorem 2.19.

(b) Since \((A^\tau_1)^\tau_2\) is \(\tau_2\) regularly open w.r.t. \(\tau_1\) and \((A^\tau_1)^\tau_2 \cap A = A^\tau_1 \cap A = A^\tau_1\), the result follows by virtue of Theorem 2.19.

**THEOREM 2.21.** If \((X, \tau_1, \tau_2)\) is \(\tau_i\) regular w.r.t. \(\tau_j\) and \(\tau_i\) \(S\)-closed w.r.t. \(\tau_j\), then \((X, \tau_i)\) is compact, where \(i, j = 1, 2; i \neq j\).

**Proof** By virtue of Theorem 1.5(a), we see that every \(\tau_i\) \(S\)-closed space w.r.t. \(\tau_j\) is \(\tau_i\) almost compact w.r.t. \(\tau_j\). Hence by Theorem 1.5(b) the result follows.

In Theorem 3.7 we shall prove a partial converse of the above theorem.

3. **PAIRWISE EXTREMALLY DISCONNECTEDNESS AND \(S\)-CLOSED SPACE.**

**DEFINITION 3.1.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(\tau_i\) extremally disconnected w.r.t. \(\tau_j\) if and only if for every \(\tau_i\) open set \(A\) of \(X\),
is \( \tau_i \) open, where \( i, j = 1,2 \) and \( i \neq j \). \( X \) is called pairwise extremally disconnected if and only if it is \( \tau_1 \) extremally disconnected w.r.t. \( \tau_2 \) and \( \tau_2 \) extremally disconnected w.r.t. \( \tau_1 \).

Datta in [8] has defined pairwise extremally disconnected bitopological space identically as above, we shall show (see Corollary 3.4) that the concept can be defined by a weaker condition.

The conclusion of the following theorem was also derived in [8] under the hypothesis that the space is pairwise Hausdorff and pairwise extremally disconnected. We prove a much stronger result here.

**THEOREM 3.2.** Let \( (X, \tau_1, \tau_2) \) be \( \tau_1 \) extremally disconnected w.r.t. \( \tau_2 \) or \( \tau_2 \) extremally disconnected w.r.t. \( \tau_1 \). Then for every pair of disjoint sets \( A, B \) in \( X \), where \( A \in \tau_1 \) and \( B \in \tau_2 \), one has \( A^{\tau_2} \cap B^{\tau_1} = \emptyset \).

**PROOF:** Suppose \( (X, \tau_1, \tau_2) \) is \( \tau_1 \) extremally disconnected w.r.t. \( \tau_2 \) and \( A \in \tau_1 \), \( B \in \tau_2 \) with \( A \cap B = \emptyset \). Then \( A^{\tau_2} \cap B = \emptyset \) ... (1). Now, if \( A^{\tau_2} \cap B^{\tau_1} \neq \emptyset \), then there exists \( x \in B^{\tau_1} \) and \( x \in A^{\tau_2} \in \tau_1 \). Hence \( A^{\tau_2} \cap B \neq \emptyset \) contradicting (1). Similarly the other case can be handled.

We prove a stronger converse of the above theorem.

**THEOREM 3.3.** \( (X, \tau_1, \tau_2) \) is pairwise extremally disconnected if for every pair of disjoint sets \( A \) and \( B \), where \( A \in \tau_1 \) and \( B \in \tau_2 \), \( A^{\tau_2} \cap B^{\tau_1} = \emptyset \) holds.

**PROOF:** Suppose \( (X, \tau_1, \tau_2) \) is not \( \tau_1 \) extremally disconnected w.r.t. \( \tau_2 \). Then there is a \( \tau_1 \) open set \( A \) such that \( A^{\tau_2} \tau_1 \). Then \( X - A^{\tau_2} \tau_2 \) and \( A \in \tau_1 \) such that \( A \cap (X - A^{\tau_2}) = \emptyset \). Hence by hypothesis, \( A^{\tau_2} \cap (X - A^{\tau_2})^{\tau_1} = \emptyset \). Then \( (X - A^{\tau_2})^{\tau_1} = X - A^{\tau_2} \) and \( X - A^{\tau_2} \) is \( \tau_1 \) closed. Thus \( A^{\tau_2} \) is \( \tau_1 \) -open. A contradiction.

Similarly, \( (X, \tau_1, \tau_2) \) is \( \tau_2 \) extremally disconnected w.r.t. \( \tau_1 \).

From Theorems 3.2 and 3.3 we have,

**COROLLARY 3.4.** \( (X, \tau_1, \tau_2) \) is pairwise extremally disconnected if and only if it is either \( \tau_1 \) extremally disconnected w.r.t. \( \tau_2 \) or \( \tau_2 \) extremally disconnected w.r.t. \( \tau_1 \).

**LEMMA 3.5.** If \( (X, \tau_1, \tau_2) \) is pairwise extremally disconnected, then for every \( \tau_1 \)
PAIRWISE S-CLOSED BITOPOLITICAL SPACES

For every s.o. set \( V \) w.r.t. \( \tau_2 \), \( V_{\tau_2}(\tau_1) = \overline{V}^{\tau_2} \) and for every s.o. set \( U \) w.r.t. \( \tau_1 \),
\[ U_{\tau_1}(\tau_2) = \overline{U}^{\tau_1} \]

**Proof:** Obviously, \( V_{\tau_2}(\tau_1) \subseteq \overline{V}^{\tau_2} \).

Now, if \( x \notin \overline{V}^{\tau_2}(\tau_1) \), then there exists a s.o. set \( W \) w.r.t. \( \tau_1 \), containing \( x \) such that \( V \cap W = \emptyset \). Then \( \overline{V}^{\tau_1} \) and \( \overline{W}^{\tau_2} \) are nonempty disjoint sets, respectively \( \tau_1 \) open and \( \tau_2 \) open. Since \( (X, \tau_1, \tau_2) \) is pairwise extremally disconnected, we have
\[ \overline{V}^{\tau_1} \cap \overline{W}^{\tau_2} = \emptyset, \text{ i.e., } \overline{V}^{\tau_2} \cap \overline{W}^{\tau_1} = \emptyset \]
Thus \( x \notin \overline{V}^{\tau_2} \). Hence \( V_{\tau_2}(\tau_1) = \overline{V}^{\tau_2} \).

Similarly the other part can be proved.

**Lemma 3.6.** In a pairwise extremally disconnected space \( (X, \tau_1, \tau_2) \), every \( \tau_1 \) regularly open set w.r.t. \( \tau_2 \) is \( \tau_1 \) open and \( \tau_2 \) closed, where \( i, j = 1, 2 \) and \( i \neq j \).

**Proof:** Let \( A \) be a \( \tau_1 \) regularly open set in \( X \) w.r.t. \( \tau_2 \) so that \( (\overline{A}^{\tau_2})^i_1 = A \).

Now, \( (X - \overline{A}^{\tau_2}) \) and \( A \) are disjoint sets, respectively \( \tau_2 \) open and \( \tau_1 \) open.

Since \( (X, \tau_1, \tau_2) \) is pairwise extremally disconnected, we have
\[ (X - \overline{A}^{\tau_2}) \cap \overline{A}^{\tau_2} = \emptyset, \text{ by Theorem 3.2.} \]
Then \( (X - \overline{A}^{\tau_2})^{i_1} = X - \overline{A}^{\tau_2} \) and \( X - \overline{A}^{\tau_2} \) is \( \tau_1 \) closed. Hence \( \overline{A}^{\tau_2} \) is \( \tau_1 \)-open, so that \( \overline{A}^{\tau_2} = (\overline{A}^{\tau_2})^{i_1} = A \) is \( \tau_1 \) open and \( \tau_2 \) closed.

Similarly, we can show that every \( \tau_2 \) regularly open set in \( X \) w.r.t. \( \tau_1 \) is \( \tau_2 \) -open and \( \tau_1 \) -closed.

**Theorem 3.7.** If \( (X, \tau_1, \tau_2) \) is pairwise extremally disconnected and \( (X, \tau_1) \) is compact, then \( (X, \tau_1, \tau_2) \) is \( \tau_1 \) S-closed w.r.t. \( \tau_2 \).

**Proof:** Let \( \{V_\alpha : \alpha \in I\} \) be a cover of \( X \) by sets that are \( \tau_1 \) s.o.w.r.t. \( \tau_2 \).

For each \( x \in X \), there is a \( V_\alpha \) containing \( x \), for some \( \alpha_x \in I \). Then there exists a \( \tau_1 \) open set \( O_\alpha \) such that \( O_\alpha \subseteq V_\alpha \subseteq \overline{O_\alpha}^{\tau_2} \). Since \( X \) is pairwise extremally disconnected, \( \overline{O_\alpha}^{\tau_2} \) is \( \tau_1 \) open for each \( x \in X \). By compactness of \( (X, \tau_1) \) there exists a finite set of points \( x_1, x_2, \ldots, x_n \) of \( X \) such that \( X = \bigcup_{k=1}^{n} (\overline{O_\alpha}^{\tau_2}) \). But \( O_\alpha \subseteq V_\alpha \), for each \( x \). Hence \( \overline{O_\alpha}^{\tau_2} \subseteq \overline{V_\alpha}^{\tau_2} \).

Hence \( X = \bigcup_{k=1}^{n} (\overline{V_\alpha}^{\tau_2}) \) and \( X \) is \( \tau_1 \) S-closed w.r.t. \( \tau_2 \).
We have earlier observed that every $\tau_i$ S-closed space $(X, \tau_1, \tau_2)$ w.r.t. $\tau_j$ is always $\tau_j$ almost compact w.r.t. $\tau_j$ for $i, j = 1, 2$ and $i \neq j$. Now we have:

**THEOREM 3.8.** If $(X, \tau_1, \tau_2)$ is $\tau_1$ almost compact w.r.t. $\tau_2$ and pairwise extremally disconnected, then $(X, \tau_1, \tau_2)$ is $\tau_1$ S-closed w.r.t. $\tau_2$.

**PROOF:** Let us consider a cover $\{V_a : a \in I\}$ of $X$ with sets that are $\tau_1$ s.o.w.r.t. $\tau_2$. For each $a \in I$, we consider the set $U_a = (V_a^{\tau_2})^{\tau_1}$ which is $\tau_1$ regularly open w.r.t $\tau_2$. Then $U_a \subseteq U_a \cup V_a \subseteq V_a^{\tau_2} = (V_a^{\tau_2})^{\tau_1} = U_a^{\tau_2}$. Since $U_a$ is $\tau_1$ regularly open w.r.t $\tau_2$, by Lemma 3.6, $U_a$ is $\tau_2$-closed and hence, $U_a \subseteq U_a \cup V_a \subseteq U_a^{\tau_2} = U_a$. Thus $U_a = U_a \cup V_a$. Again, $U_a$ being $\tau_1$-open, for each $a \in I$, it follows that $(U_a \cup V_a : a \in I)$ is a $\tau_1$-open cover of $(X, \tau_1, \tau_2)$. $(X, \tau_1, \tau_2)$ being $\tau_1$ almost compact w.r.t. $\tau_2$, there exists a finite subfamily $I_0$ of $I$ such that $X = \bigcup_{a \in I_0} (U_a \cup V_a)^{\tau_2}$. Now, since $U_a \cup V_a \subseteq V_a^{\tau_2}$ for each $a \in I$ and hence $X \subseteq \bigcup_{a \in I_0} (V_a^{\tau_2})$.

**Hence $(X, \tau_1, \tau_2)$ is $\tau_1$ S-closed w.r.t. $\tau_2$.**

4. **SEMI CONTINUITY, IRRESOLUTE FUNCTIONS AND S-CLOSEDNESS.**

**DEFINITION 4.1.** [7] A function $f$ from a bitopological space $(X, \tau_1, \tau_2)$ into a bitopological space $(Y, \sigma_1, \sigma_2)$ is called $\tau_1 \sigma_1$ semi-continuous w.r.t. $\tau_2$ if for each $A \in \sigma_1$, $f^{-1}(A)$ is $\tau_1$ s.o.w.r.t. $\tau_2$. Similar goes the definition of $\tau_2 \sigma_2$ semi-continuity of $f$ w.r.t. $\tau_1$. $f$ is called pairwise semi-continuous if $f$ is $\tau_1 \sigma_1$ semi-continuous w.r.t. $\tau_2$ and $\tau_2 \sigma_2$ semi-continuous w.r.t. $\tau_1$.

**LEMMA 4.2.** If a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\tau_1 \sigma_1$ semi-continuous w.r.t. $\tau_2$, then for any subset $A$ of $X$, $f(A) \cap (\tau_1(\tau_2)) \subseteq f(A)^{\sigma_1}$.

**PROOF:** Let $y \in f(A \cap (\tau_1(\tau_2)))$ and $y \in V \in \sigma_1$. Then there exists $x \in A \cap (\tau_1(\tau_2))$ such that $f(x) = y$ and $x \in f^{-1}(V)$ and $f^{-1}(V)$ is $\tau_1$ s.o.w.r.t. $\tau_2$. Hence $f^{-1}(V) \cap A \neq \emptyset \Rightarrow f(f^{-1}(V) \cap A) \neq \emptyset \Rightarrow V \cap f(A) \neq \emptyset \Rightarrow y \in f(A)^{\sigma_1}$.

**THEOREM 4.3.** Pairwise semi-continuous surjection of a pairwise S-closed space onto a pairwise Hausdorff space is pairwise H-closed.

**PROOF:** Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise semi-continuous surjection, where $X$ is pairwise S-closed. We first show that $(Y, \sigma_1, \sigma_2)$ is $\sigma_1$ almost compact w.r.t. $\sigma_2$. Let $\{V_a : a \in I\}$ be a $\sigma_1$ open cover of $Y$. Then
PAIRWISE S-CLOSED BITOPOLITICAL SPACES

\( \{ f^{-1}(V_a) : \alpha \in \Pi \} \) is a cover of \( X \) by sets that are \( \tau_1 \) s.o.w.r.t. \( \tau_2 \). Since \( X \) is \( \tau_1 \) S-closed w.r.t. \( \tau_2 \), there exists a finite subfamily \( I_0 \) of \( I \), such that

\[ X = \bigcup_{\alpha \in I_0} f^{-1}(V_a) \].

We show that \( \bigcup_{\alpha \in I_0} f^{-1}(V_a)_{\tau_2}(\tau_1) = X \). In fact, let \( x \in X \)

and \( W \) be any \( \tau_2 \) s.o. set w.r.t. \( \tau_2 \), containing \( x \). Then there exists \( U \in \tau_2 \) such that \( U \subset W \subset U^{1} \) and \( U \neq \emptyset \). Since \( \bigcup_{\alpha \in I_0} f^{-1}(V_a) \) is \( \tau_2 \) dense in \( X \), every nonempty \( \tau_2 \) open set must intersect \( \bigcup_{\alpha \in I_0} f^{-1}(V_a) \) and hence

\[ x \in \bigcap_{\alpha \in I_0} f^{-1}(V_a) \].

Now,

\[ Y = f(X) = f \left( \bigcup_{\alpha \in I_0} f^{-1}(V_a) \right) \tau_1(\tau_1) \]

\[ \subseteq f \left( \bigcup_{\alpha \in I_0} f^{-1}(V_a) \right) \tau_2(\tau_1) \]

\[ = \bigcup_{\alpha \in I_0} \overset{\text{f}}{V_a} \).

(Using Lemma 4.2 and the fact that \( f \) is \( \tau_2 \) \( \sigma_2 \) semi-continuous w.r.t. \( \tau_1 \)). Thus by Theorem 1.5(a), \( Y \) is \( \sigma_1 \) almost compact w.r.t. \( \sigma_2 \). Similarly, \( Y \) is \( \sigma_2 \) almost compact w.r.t. \( \sigma_1 \). Since \( Y \) is pairwise Hausdorff, it finally follows by virtue of Theorem 1.5(c) that \( (Y, \sigma_1, \sigma_2) \) is pairwise H-closed.

DEFINITION 4.4. A function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called \( \tau_1 \sigma_1 \sigma_2 \) irresolute w.r.t \( \tau_2 \) if for every \( \sigma_1 \) s.o. set \( V \) w.r.t. \( \sigma_2 \), \( f^{-1}(V) \) is \( \tau_1 \) s.o.w.r.t. \( \tau_2 \). Functions that are \( \tau_2 \sigma_2 \) irresolute w.r.t. \( \tau_1 \) and pairwise irresolute can be defined in the usual manner.

Clearly, every \( \tau_1 \sigma_1 \) irresolute function w.r.t \( \tau_j \) is \( \tau_i \sigma_i \) semi-continuous w.r.t. \( \tau_j \), where \( i, j = 1, 2 \) but \( i \neq j \), but it can be shown that the converse is not true, in general. This converse is true if the function \( f \) is, in addition, pairwise open [7].

LEMMA 4.5. A function \( f \) from a bitopological space \((X, \tau_1, \tau_2)\) to a bitopological space \((Y, \sigma_1, \sigma_2)\) is \( \tau_1 \sigma_1 \) irresolute w.r.t \( \tau_2 \) if and only if for every subset \( A \) of \( X \), \( f(A)_{\tau_1(\tau_2)} \subset f(A)_{\sigma_1(\sigma_2)} \).

PROOF: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be \( \tau_1 \sigma_1 \) irresolute w.r.t \( \tau_2 \) and \( A \subset X \). Then \( f^{-1}(f(A))_{\sigma_1(\sigma_2)} \) is \( \tau_1 \) s.c.l.w.r.t. \( \tau_2 \). Since \( A \subset f^{-1}(f(A)) \subset f^{-1}(f(A)_{\sigma_1(\sigma_2)}) \), we have \( A_{\tau_1(\tau_2)} \subset f^{-1}(f(A)_{\sigma_1(\sigma_2)}) \) and hence
f(A_{\tau_1(\tau_2)}) = f^{-1}(f(A)_{\sigma_1(\sigma_2)})$, i.e. $f(A_{\tau_1(\tau_2)}) \subseteq f(A)_{\sigma_1(\sigma_2)}$.

Conversely, let $B$ be $\sigma_1$ s.c.l.w.r.t. $\sigma_2$ in $Y$. By hypothesis, $f(f^{-1}(B)_{\tau_1(\tau_2)}) \subseteq f^{-1}(B)_{\sigma_1(\sigma_2)} \subseteq B_{\sigma_1(\sigma_2)} = B$.

Then $f^{-1}(B)_{\sigma_1(\tau_2)} \subseteq f^{-1}(B)$ and hence $f^{-1}(B) = f^{-1}(B)_{\tau_1(\tau_2)}$. This shows that $f^{-1}(B)$ is $\tau_1$ s.c.l.w.r.t. $\tau_2$ and then $f$ is $\tau_1 \sigma_1$ irresolute w.r.t. $\tau_2$.

**COROLLARY 4.6.** If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\tau_1 \sigma_1$ irresolute w.r.t. $\tau_j$, then for any subset $A$ of $X$, $f(A)_{\tau_1(\tau_j)} \subseteq f(A)_{\sigma_1(\sigma_j)}$, where $i, j = 1, 2$ and $i \neq j$.

**PROOF:** For every subset $B$ of a bitopological space $(X, \tau_1, \tau_2)$ we always have $B_{\tau_1(\tau_j)} \subseteq B_{\tau_1(\tau_j)}$, for $i, j = 1, 2$ and $i \neq j$. Hence by Lemma 4.5, the corollary follows.

**NOTE 4.7.** Following a similar line of proof as in Lemma 4.2, we could also prove the above corollary 4.6.

**THEOREM 4.8.** Let $(X, \tau_1, \tau_2)$ be pairwise extremally disconnected and $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be pairwise irresolute, where $(Y, \sigma_1, \sigma_2)$ is a bitopological space. If a subset $G$ of $X$ is pairwise $S$-closed in $X$, then $f(G)$ is pairwise $S$-closed in $Y$.

**PROOF:** Let $(A_\alpha: \alpha \in I)$ be a cover of $f(G)$ by sets that are $\sigma_1$ s.w.r.t. $\sigma_2$ in $Y$. Then $f^{-1}(A_\alpha)$ is $\tau_1$ s.w.r.t. $\tau_2$ in $X$, for each $\alpha \in I$ and $(f^{-1}(A_\alpha): \alpha \in I)$ is a cover of $G$. Since $G$ is pairwise $S$-closed in $X$, there exist a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $G \subseteq \bigcup_{k=1}^{n} f^{-1}(A_{\alpha_k})_{\tau_2}$. By Lemma 3.5, we have $f^{-1}(A_{\alpha_k})_{\tau_2} = f^{-1}(A_{\alpha_k}_{\tau_1})_{\tau_2}$ for $k = 1, 2, \ldots, n$. Since $f$ is $\tau_2 \sigma_2$ irresolute w.r.t. $\tau_1$, we have by Lemma 4.5 $f(f^{-1}(A_{\alpha_k})_{\tau_2} \tau_1) \subseteq f^{-1}(A_{\alpha_k})_{\tau_2} \subseteq A_{\alpha_k}^\sigma_{\alpha_k} \subseteq A_{\alpha_k}^\sigma_{\alpha_k} \subseteq A_{\alpha_k}^\sigma_{\alpha_k}$ for $k = 1, 2, \ldots, n$.

Hence $f(G) \subseteq f\left(\bigcup_{k=1}^{n} f^{-1}(A_{\alpha_k})_{\tau_2}\right) \subseteq \bigcup_{k=1}^{n} A_{\alpha_k}^\sigma_{\alpha_k}$ and then $f(G)$ is $\sigma_1$ s-closed w.r.t. $\sigma_2$ in $Y$. Similarly, $f(G)$ is $\sigma_2$ s-closed w.r.t. $\sigma_1$ in $Y$. Hence $f(G)$ is pairwise $S$-closed in $Y$. This completes the proof.
NOTE 4.9. If the set \( G \) of Theorem 4.8 is the whole space \( X \), then we do not require the condition that \((X, \tau_1, \tau_2)\) is pairwise extremally disconnected. In fact, proceeding in a similar fashion as in Theorem 4.3 and using Corollary 4.6, we can have:

**THEOREM 4.10.** If \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise irresolute and surjective, where \((X, \tau_1, \tau_2)\) is pairwise \( S \)-closed, then \((Y, \sigma_1, \sigma_2)\) is also pairwise \( S \)-closed.

**THEOREM 4.11.** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be \( \tau_1 \sigma_1 \) semi-continuous w.r.t. \( \sigma_2 \), \( f: (X, \tau_2) \rightarrow (Y, \sigma_2) \) is continuous and open. If \( G \subseteq X \) is \( \tau_1 \) \( S \)-closed w.r.t. \( \tau_2 \) in \( X \), then \( f(G) \) is \( \sigma_1 \) \( S \)-closed w.r.t. \( \sigma_2 \) in \( Y \).

**PROOF:** Let \( \{ U_\alpha : \alpha \in I \} \) be a cover of \( f(G) \) by sets that are \( \sigma_1 \) s.o.w.r.t. \( \sigma_2 \).

For each \( \alpha \), there is \( V_\alpha \subseteq \sigma_1 \) such that \( V_\alpha \subseteq U_\alpha \subseteq V_\alpha^{\sigma_2} \). Since \( f: (X, \tau_2) \rightarrow (Y, \sigma_2) \) is open, we have \( f^{-1}(V^{\sigma_2}_\alpha) \subseteq f^{-1}(V_\alpha) \). Since \( f \) is \( \tau_1 \sigma_1 \) semi-continuous w.r.t. \( \tau_2 \), \( f^{-1}(V_\alpha) \) is \( \tau_1 \) s.o.w.r.t. \( \tau_2 \) and hence there exists \( 0 \in \tau_1 \), such that

\[
0 \subseteq f^{-1}(V_\alpha) \subseteq 0^{\tau_2} \Rightarrow 0 \subseteq f^{-1}(V_\alpha) \subseteq 0^{\tau_2}. \quad \text{Thus,} \quad 0 \subseteq f^{-1}(V_\alpha) \subseteq f^{-1}(V^{\sigma_2}_\alpha) \subseteq f^{-1}(V_\alpha^{\sigma_2}) \subseteq f^{-1}(V^{\sigma_2}_\alpha) \subseteq 0^{\tau_2}. \quad \text{That is,} \quad 0 \subseteq f^{-1}(U_\alpha) \subseteq 0^{\tau_2} \quad \text{and} \quad 0 \in \tau_1. \quad \text{Therefore,}
\]

\[
f^{-1}(U_\alpha) \text{ is } \tau_1 \text{ s.o.w.r.t. } \tau_2, \text{ for each } \alpha \in I, \text{ and } \{ f^{-1}(U_\alpha) : \alpha \in I \} \text{ is a cover of } G. \text{ Then there exists a finite number of indices } \alpha_1, \ldots, \alpha_n \text{ such that}
\]

\[
G \subseteq \bigcup_{i=1}^{n} f^{-1}(U^{\tau_2}_{\alpha_i}) \quad \text{Since } f: (X, \tau_2) \rightarrow (Y, \sigma_2) \text{ is continuous,}
\]

\[
f \left[ f^{-1}(U^{\tau_2}_{\alpha_i}) \right] \subseteq \bigcup_{i=1}^{n} U^{\sigma_2}_{\alpha_i}, \text{ for } i = 1, 2, \ldots, n. \quad \text{Therefore, } f(G) \subseteq \bigcup_{i=1}^{n} U^{\sigma_2}_{\alpha_i}, \text{ and then}
\]

\[
f(G) \text{ is } \sigma_1 \text{ S-closed w.r.t. } \sigma_2 \text{ in } Y.
\]

**COROLLARY 4.12.** Pairwise \( S \)-closedness is a bitopological invariant.

**PROOF:** Since every pairwise continuous function is pairwise semi-continuous, the corollary follows by virtue of Theorem 4.11.

**COROLLARY 4.13.** Let \( \{ (X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in I \} \) be a family of bitopological spaces and \( (X, \tau_1, \tau_2) \) be their product space. If \((X, \tau_1, \tau_2)\) is pairwise \( S \)-closed, then each \((X_\alpha, \tau_1^\alpha, \tau_2^\alpha)\) is also pairwise \( S \)-closed.

**PROOF:** Since \( P_\alpha : (X, \tau_1) \rightarrow (X_\alpha, \tau_1^\alpha) \) is an open, continuous surjection, for \( i = 1, 2 \) and for each \( \alpha \in I, \), the corollary becomes evident because of Theorem 4.11.
THEOREM 4.14. The pairwise irresolute image of a pairwise S-closed and pairwise extremally disconnected bitopological space in any pairwise Hausdorff bitopological space is pairwise closed.

PROOF: Let $f$ be a pairwise irresolute function from a pairwise S-closed and pairwise extremally disconnected space $(X, \tau_1, \tau_2)$ into a pairwise Hausdorff space $(Y, \sigma_1, \sigma_2)$. Let $y \in f(X)$ and $N_1(y)$ denote the $\sigma_1$-open neighborhood system at $y$ in $(Y, \sigma_1, \sigma_2)$. Then $F = \{f^{-1}(V): V \in N_1(y)\}$ is a filter-base in $X$.

Since $X$ is $\tau_2$ S-closed w.r.t. $\tau_1$, $F$ has a $\tau_2$ S-accumulation point $x$ w.r.t. $\tau_1$.

We show that $f(F)$ has $f(x)$ as a $\sigma_2$ accumulation point. In fact, let $f(x) \in V \in \sigma_2$. Then $f^{-1}(V)$ is $\tau_2$ s.o.w.r.t. $\tau_1$ and contains $x$. Now, for each $W \in N_1(y), f^{-1}(W) \in F$ and hence $f^{-1}(W) \cap \overline{f^{-1}(V)}^{\tau_1} \neq \emptyset$. Since $(X, \tau_1, \tau_2)$ is pairwise extremally disconnected, we then must have $\overline{f^{-1}(W)}^{\tau_1} \cap \overline{f^{-1}(V)}^{\tau_2} \neq \emptyset$.

Indeed, if $\overline{f^{-1}(W)}^{\tau_1} \cap \overline{f^{-1}(V)}^{\tau_2} = \emptyset$, then $\overline{f^{-1}(W)}^{\tau_1} \cap \overline{f^{-1}(V)}^{\tau_2} = \emptyset$, i.e., $f^{-1}(W)^{\tau_2} \cap \overline{f^{-1}(V)}^{\tau_1} = \emptyset$ which is not the case.

Now, $\emptyset \neq f(\overline{f^{-1}(W)}^{\tau_1} \cap \overline{f^{-1}(V)}^{\tau_2}) \subset f(\overline{f^{-1}(W)}^{\tau_1} \cap \overline{f^{-1}(V)}^{\tau_2}) \subset W \cap V$. Hence $W \cap V \neq \emptyset$. This shows that $f(x)$ is a $\sigma_2$ accumulation point of $f(F)$ in $Y$. But $f(F)$ being finer than $N_1(y), N_1(y)$ also $\sigma_2$ accumulates to $f(x)$. Now, if $y \neq f(x)$, by pairwise Hausdorff property of $(y, \sigma_1, \sigma_2)$, there exist $\sigma_1$ open set $A$ and $\sigma_2$ open set $B$ such that $y \in A, f(x) \in B$ and $A \cap B = \emptyset$. Since $A \in N_1(y), f(f^{-1}(A)) \in f(F)$ and hence $B \cap f(f^{-1}(A)) \neq \emptyset$, because $f(x)$ is a $\sigma_2$ accumulation point of $f(F)$. In other words $B \cap A \neq \emptyset$, which is a contradiction. Hence $y = f(x)$ and then $f(y) = f(x)$. Consequently $f(x)$ is $\sigma_2$ closed in $Y$. Similarly $f(x)$ is $\sigma_1$ closed in $Y$. This completes the proof.

ACKNOWLEDGEMENT. I sincerely thank Dr. S. Ganguly, Reader, Department of Pure Mathematics, Calcutta University, for his kind help in the preparation of this paper.

REFERENCES


