INTEGRAL OPERATORS OF CERTAIN UNIVALENT FUNCTIONS

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ABSTRACT. A function $f$, analytic in the unit disc $A$, is said to be in the family $R_n(\alpha)$ if \( \text{Re}\{(z^nf(z))^{n+1}/(z^{n-1}f(z))^n\} > (n+\alpha)/(n+1) \) for some \( \alpha (0 < \alpha < 1) \) and for all $z$ in $A$, where $n \in \mathbb{N}$, $\mathbb{N} = \{0,1,2,\ldots\}$. The class $R_n(\alpha)$ contains the starlike functions of order $\alpha$ for $n \geq 0$, and the convex functions of order $\alpha$ for $n \geq 1$. We study a class of integral operators defined on $R_n(\alpha)$. Finally an argument theorem is proved.

KEY WORDS AND PHRASES: Univalent, convolution, starlike, convex


INTRODUCTION.

Let $A$ denote the family of functions $f$ which are analytic in the unit disc $A = \{z: |z| < 1\}$ and normalised such that $f(0) = 0 = f'(0) - 1$. The Hadamard product or convolution of two functions $f, g \in A$ is denoted by $f \ast g$. Let $D^n f = (z/(1-z)^{n+1}) f, n \in \mathbb{N} = \{0,1,2,\ldots\}$ which implies that $D^n f = z(z^{n-1}f(n))/n!, n \in \mathbb{N}$.

Denote by $S^\alpha(\alpha)$ and $K(\alpha)$ the subfamilies of $A$ whose members are, respectively, starlike of order $\alpha$ and convex of order $\alpha$, $0 < \alpha < 1$. Then

\[
 f \in S^\alpha(\alpha) \iff \text{Re}(D^2 f/Df) > \alpha, z \in A,
\]

\[
 f \in K(\alpha) \iff \text{Re}(D^2 f/Df) > (1+\alpha)/2, z \in A.
\]

Ruscheweyh [16] introduced the classes $\{K_n\}$ of functions $f \in A$ which satisfy the condition

\[
 \text{Re}(D^{n+1} f/Df) > \frac{1}{n}, z \in A \tag{1.1}
\]

so that the definition of $K_n$ is a natural extension of $S^\alpha(1/2)$, and $K(0)$. He proved that $K_{n+1} \subseteq K_n$ for each $n \in \mathbb{N}_0$. Since $K_0 = S^\alpha(1/2)$, the elements of $K_n$ are univalent and starlike of order $1/2$.

In this paper, we consider the classes of functions $f \in A$ which
satisfy the condition
\[ Re\left( z^{n}f' / f^n \right) > \alpha, \; z \in \Delta \] (1.2)
for some \( \alpha (0 \leq \alpha < 1) \). We denote these classes by \( R_n(\alpha) \). We have \( R_n(\alpha) = S^*(\alpha) \) and \( R_n(\alpha) = K(\alpha) \) for \( 0 \leq \alpha < 1 \). The classes \( R_n = R_n(0) \) were considered earlier by Singh and Singh [17]. It is readily seen that for each \( n \geq 1 \), \( R_n(\alpha) \subseteq R_0(\alpha) = R(\alpha) \) and for each \( n \geq 1 \), \( R_n(\alpha) \subseteq K_n \). We note that in definition (1.2), restriction \( \alpha \geq 0 \) can be replaced by \( \alpha \geq (1-n)/2 \) for each \( n \geq 1 \) and, further, that the negative choices of \( \alpha \) permit us fully to partition \( K_n \) into classes \( R_n(\alpha) \subseteq K_n \) such that
\[ \cup R_n(\alpha) = K_n \]
\[ \frac{1-n}{2} \leq \alpha < 1 \]

It can be easily seen that \( R_{n+1}(\alpha) \subseteq R_n(\alpha) \) for each \( n \in \mathbb{N}_0 \) and for all \( \alpha \). These inclusion relations establish that \( R_n(\alpha) \subseteq S^*(\alpha) \) for each \( n \geq 0 \) and \( R_n(\alpha) \subseteq K(\alpha) \) for each \( n \geq 1 \).

An important problem in univalent functions is the following: Given a compact family \( F \) and an operator \( J \) defined on \( F \), is \( J(f) \in F \) for every \( f \in F \) ? Libera [11] established that the operator
\[ J(f) = \frac{2}{z} \int_{0}^{z} f(t) dt \] (1.3)
preserves convexity, starlikeness, and close-to-convexity. Bernardi [5] greatly generalised Libera's results. Many authors [1,2,7,8,12,15,17] studied operators of the form
\[ J(f) = \frac{1+\gamma}{2} \int_{0}^{z} t^{\gamma-1} f(t) dt \] (1.4)
where \( \gamma \) is a real (or complex) constant and \( f \) belongs to some favoured class of univalent functions from \( A \). Recently, operators (1.4) have been studied in more general form by Causey and White [6], Miller, Mocanu and Reade [14], Barnard and Kellogg [3], and Bajpai [2]

In this paper, we study a class of integral operators of the form (1.4) defined on our family \( R_n(\alpha) \). We also obtain an argument theorem for the class \( R_n(\alpha) \).

2. INTEGRAL OPERATORS.

Let \( \gamma \) be a complex number with \( \text{Re} \gamma > 1 \). We define \( h_\gamma \) by
Let the operator $J: A ightarrow A$ be defined by $F = J(f)$, where

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt$$

Then the function $F$ can also be written in the form

$$F(z) = f(z)h_\gamma(z).$$

We need the following result of Jack [9] which is also due to Suffridge [18]

**Lemma.** Let $w$ be nonconstant and analytic in $|z| < r < 1$, $w(0) = 0$ If $|w|$ attains its maximum value on the circle $|z| = r$ at $z_0$, then

$$z_0w'(z_0) = kW(z_0),$$

where $k$ is a real number and $k \geq 1$

We first give a condition on $f \in A$ for which the function $J(f)$ belongs to $R_n(a)$

**Theorem I.** Let $0 < a < 1$, and $\gamma \neq -1$ be a complex constant such that $\Re \gamma < a$, $\Im \gamma \geq 0$, and $|\gamma|^2 + 2\alpha(1 + \Re \gamma) \geq 1$. If for a given $n \in N_0$, $f \in A$ satisfies the condition

$$\Re \frac{z(D^nf(z))'}{D^nf(z)} > a - \frac{(1-a)(1+\Re \gamma)}{2(|\gamma|^2 + 2\alpha \Re \gamma + \alpha^2 + (1-a)\Im \gamma)}$$

for all $z \in \Delta$, then $F(z)$ given by (2.2) belongs to $R_n(a)$.

**Proof.** From (2.2), we obtain

$$z(D^nF(z))' + \gamma D^nF(z) = (\gamma + 1)D^n\phi(z).$$

(2.4)

Define $w$ in $\Delta$ by

$$\frac{z(D^nF(z))'}{D^nF(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)}.$$

(2.5)

Here $\phi(z)$ is analytic in $\Delta$ with $\phi(0) = 0$ and $\phi(z) \neq -1$, $z \in \Delta$

We need to show that $|w(z)| < 1$ for all $z \in \Delta$. In view of (2.4), (2.5) yields

$$\frac{D^n\phi(z)}{D^nF(z)} = \frac{(1+\gamma)+(2\alpha-1+\gamma)w(z)}{(1+\gamma)(1+w(z))}$$

(2.6)

Differentiating (2.6) logarithmically and simplifying, we obtain
\[
\frac{z(D^n f(z))'}{D^n f(z)} = \alpha + (1-\alpha) \frac{1-w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))(1+y+(2\alpha-1+y)w(z))} \tag{2.7}
\]

Now (2.7) should yield \(|w(z)| < 1\) for all \(z \in \Delta\) for otherwise, there exists a point \(z_0 \in \Delta\) at which \(|w(z_0)| = 1\) and by Lemma, we have

\[z_0 w'(z) = kw(z_0), \quad k \geq 1.\]

For this value of \(z = z_0\), we find that (2.7) yields

\[
\Re \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} = \alpha - \frac{2k(1-\alpha)(\alpha + \Re y)}{|(1+y)+(2\alpha-1+y)w(z_0)|^2} \tag{2.8}
\]

which contradicts (2.3) Hence \(|w(z)| < 1\) for all \(z \in \Delta\) and by (2.5), it follows that \(F(z) \in R_n(\alpha)\).

COROLLARY. If for a given \(n \in N_0\), \(f \in \mathcal{A}\) satisfies the condition

\[
\Re \frac{z(D^n f(z))'}{D^n f(z)} > \frac{2\alpha(y+\alpha)-(1-\alpha)}{2(y+\alpha)}, \quad z \in \Delta, \tag{2.9}
\]

where \((\alpha, \gamma)\) is any point in the set

\[D = \{(\alpha, \gamma) : \gamma+2\alpha \geq 1, \quad 0 \leq \alpha < 1, \quad \gamma > -1\},\]

then \(F(z)\) given by (2.2) belongs to \(R_n(\alpha)\).

PROOF. If \(\gamma \neq -1\) is a real constant such that \(\gamma + \alpha \geq 0\), then

\[|\gamma|^2+2\alpha(1+\Re y) \geq 1\]

implies \((\gamma+1)(\gamma+2\alpha-1) \geq 0\). The result follows from Theorem 1.

It is easy to show that if \(f \in R_n(\alpha)\), then \(f\) satisfies the condition (2.3). Thus it follows from Theorem 1 that \(J(R_n(\alpha)) \subset R_n(\alpha)\). More precisely, we state the result in

THEOREM 2. If \(f \in R_n(\alpha)\), then the function

\[J(f) = \frac{Y+1}{Z} \int_0^{Z} f(t) t^{-1} dt\]

is again an element of \(R_n(\alpha)\), where \(\gamma \neq -1\) is a complex constant with restrictions as stated in Theorem 1.

REMARK 1. Letting \(n = 0 = \gamma - 1\) and \(n = 1 = \gamma\), in Theorem 1, we get \(L(S^*(B)) \subset S^*(\alpha)\) and \(L(K(\beta)) \subset K(\alpha)\) respectively, where \(L\) is the Libera transform defined in (1.3), and

\[B = ((2\alpha^2+3\alpha-1)/2(1+\alpha)) < \alpha.\]

These results improve the earlier results due to Libera [11] and Bernardi [5] in the sense that their results hold under much weaker conditions.
In [2], Bajpai has established that \( J(S^*) \subseteq S^\alpha(\alpha) \) for some \( \alpha \). We generalize this result in

**Theorem 3.** Let \( J: \mathbb{A} \to \mathbb{A} \) be defined as in (2 2), where \( \gamma \) is a complex constant. If \( f \in R_n \), then \( J(f) \in R_n(\alpha) \), where \( \alpha \) satisfies the inequality

\[
\alpha \leq 2(1-\gamma)(\alpha + \Re \gamma), \quad 0 \leq \alpha < 1
\]

**Proof.** Proceeding as in Theorem 1 and applying Lemma, we have

\[
\text{Re} \left[ \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} \right] \leq \alpha - \frac{2(1-\gamma)(\alpha + \Re \gamma)}{(1+\gamma+2\alpha-1+\gamma)w(z_0)^2}
\]

where \( \Re \gamma \geq -\alpha \). Since the right-hand side is \( \leq 0 \), we have a contradiction for \( f \in R_n \subseteq R_n(0) \). Thus we must have \( |w(z)| < 1 \) for all \( z \) in \( \Delta \) and by (2 5), it follows that \( J(f) \in R_n(\alpha) \).

**Remark 1.** If we let \( n = 0 = \gamma - 1 \) in the above theorem, then

\[
L(S^*) \subseteq S^\alpha(\gamma - \frac{\sqrt{17} - 3}{4}), \quad L(f) = (2/z) \int_0^z f(t) dt
\]

Thus we have recovered a result of Miller, Mocanu and Reade ([14], pp 162-163).

**Remark 2.** If \( n = 1 \), \( \gamma \) is a real constant such that \( \gamma + \alpha \geq 0 \), and \( f \in K \), then it follows from Theorem 3 that the function \( F(z) \) in (2 2) is an element of \( K(\alpha) \), where

\[
\alpha = \frac{-(2\gamma+1) + \sqrt{(2\gamma-1)^2 + 8(1+\gamma)}}{4}
\]

This result was proved by Miller, Mocanu and Reade ([14], pp 165). Further, this is an improvement of an earlier result due to Bernardi [5], who proved that \( f \in K \) implies \( F \in K \).

For \( \gamma = n \), where \( n \in \mathbb{N}_0 \), we have an improvement over Theorem 2

**Theorem 4.** Let

\[
F(z) = f(z) \ast h_n(z) = \frac{z^n+1}{z} \int_0^z f(t) t^{n-1} dt
\]

If \( f \in R_n(\alpha) \), then \( F \in R_{n+1}(\alpha) \)

**Proof.** From (2.10), we obtain

\[
z(D^{n+1} F(z))' + nD^{n+1} F(z) = (n+1)D^{n+1} f(z)
\]

and
Using the identity
\[ z(D^n f(z))' + nD^n f(z) = (n+1)D^{n+1} f(z) \quad (2.12) \]

in (2.11) and (2.12), we obtain
\[ (n+1)D^{n+1} f(z) = (n+2)D^{n+2} f(z) - D^{n+1} f(z) \quad (2.14) \]

and
\[ D^n f(z) = D^{n+1} f(z) \quad (2.15) \]

In view of the identity (2.13) and the relations (2.14) and (2.15),
\[ f \in R_n(\alpha) \text{ yields} \]
\[ \text{Re} \left\{ \frac{(n+2)D^{n+2} f(z) - (n+1)D^{n+1} f(z)}{D^{n+1} f(z)} \right\} > \alpha \]

which implies that
\[ \text{Re} \left\{ \frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} \right\} > \alpha , \quad z \in \Delta \]

This proves that \( f \in R_{n+1}(\alpha) \).

**REMARK** For \( n = 0 \), Theorem 4 gives the well known result:
\[ J(S^0(\alpha)) \subset K(\alpha), \text{ where } J(f) = \int_0^1 f(t)/t \, dt \]

We now investigate the converse of Theorem 2. In fact, we find the sharp radius of the disc in which \( f \in R_n(\beta) \) when \( F \), defined in (2.2), is in \( R_n(\alpha) \) for \( 0 \leq \alpha < 1, \ 0 < \beta \leq 1 \): In [12], Libera and Livingston have solved this converse problem for the case \( n = 0, \ \gamma = 1 \) when \( \alpha \leq \beta < 1 \). These authors were not able to obtain suitable results for the complementary case when \( \beta < \alpha \). However, the method used in the next theorem gives results that are more general and also covers both \( \beta \geq \alpha \) and \( \beta < \alpha \).

**THEOREM 5.** If \( F \) is an element of \( R_n(\alpha) \) for \( n \geq 0 \) and \( 0 \leq \alpha < 1 \),
\[ F(z) = \frac{1+\gamma}{z} \int_0^2 f(t)t^\gamma \, dt \quad (2.16) \]

with \( z \in \Delta, \text{Re} \gamma \geq -\alpha, \text{ and } 0 \leq \beta < 1 \), then the function \( f \) is an element of \( R_n(\beta) \) for \( |z| < r_0 \), where \( r_0 \) is the smallest positive root in \((0,1)\) of the equation
\[ (\gamma+2\alpha-1)(2\alpha-\beta-1)r^2+2((\gamma+\alpha)(\alpha-\beta)-(1-\alpha)(2-\alpha))r+(\gamma+1)(1-\beta) = 0 \quad (2.17) \]

The result is sharp.
PROOF Since \( F \in R_n(\alpha) \), we can write

\[
\frac{z(D^nF(z))'}{D^nF(z)} = \alpha + (1-\alpha)P_n(z),
\]

(2.18)

where \( P_n(z) \) is analytic in \( \Delta \) and satisfies the conditions \( P_n(0) = 1 \) \( \Re P_n(z) > 0 \) for \( z \in \Delta \). Using the identity

\[
z(D^nF(z))' = (n+1)D^{n+1}F(z) - nD^nF(z)
\]

(2.19)

in (2.18) and then taking logarithmic derivative, we obtain

\[
z(D^{n+1}F(z))' = D^{n+1}F(z)[\alpha+(1-\alpha)P_n(z) + \frac{(1-\alpha)zP'(z)}{n+\alpha+(1-\alpha)P_n(z)}]
\]

(2.20)

From (2.16) we obtain

\[
z(D^{n+1}F(z))' + \gamma D^{n+1}F(z) = (\gamma+1)D^{n+1}F(z).
\]

(2.21)

From (2.20) and (2.21) we have

\[
(\gamma+1)D^{n+1}F(z) = D^{n+1}F(z)[\alpha+\gamma+(1-\alpha)P_n(z) + \frac{(1-\alpha)zP'(z)}{n+\alpha+(1-\alpha)P_n(z)}]
\]

(2.22)

Also (2.18) together with the identity (2.4) yields

\[
(1+\gamma)D^nF(z) = D^nF(z)(\alpha+\gamma+(1-\alpha)P_n(z)).
\]

(2.23)

Now from the relations (2.22), (2.23), and (2.18) we conclude that

\[
\frac{z(D^nF(z))'}{D^nF(z)} - \beta = \alpha - \beta + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP'(z)}{\alpha+\gamma+(1-\alpha)P_n(z)}.
\]

(2.24)

Using the well known estimates

\[
|zP_n'(z)| \leq \frac{2r}{(1-r^2)}\Re P_n(z)
\]

and

\[
\Re P_n(z) \geq \frac{(1-r)}{(1+r)}, \quad |z| = r
\]

in (2.24), we obtain

\[
\Re \left\{ \frac{z(D^nF(z))'}{D^nF(z)} \right\} \geq (\alpha-\beta) + \frac{(1-\alpha)((1-r)(\gamma+\gamma+2\alpha-1)r)-2r}{(1-r)((\gamma+2\alpha-1)r+\gamma+1)}
\]

(2.25)

where \( \Re \gamma \geq -\alpha \). Therefore,

\[
\Re \left\{ \frac{z(D^nF(z))'}{D^nF(z)} \right\} > \beta
\]

if the right side of (2.25) is positive, which is satisfied provided that \( r < r_0 \), where \( r_0 \) is the smallest positive root in \((0,1)\) of (2.17).

The result in the theorem is sharp with the function \( f \) defined by

\[
f(z) = \frac{1}{(1+c)}z^{-c}(z^cF(z))',
\]

(2.26)
where \( c = \text{Re} \gamma \geq -\alpha \), and \( F \) is given by

\[
\frac{z^{(D^nF(z))'}}{D^nF(z)} = \frac{1-(2\alpha-1)z}{1-z} \tag{2.27}
\]

REMARK. By specializing choices of \( \alpha, \beta, \gamma, \) and \( n \), theorem 5 gives rise to the corresponding results obtained earlier in \([3, 4, 8, 12, 13, 15]\) and by many others.

3 AN ARGUMENT THEOREM.

THEOREM 6 If \( f \in R_n(\alpha) \), then

\[
|\arg \frac{D^k f(z)}{z} | - 2(1-\alpha) \sin^{-1} r + \sum_{m=0}^{k-1} \sin^{-1} \left( \frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r^2} \right)
\]

for each \( k(0 \leq k \leq n+1) \).

PROOF We may write

\[
\frac{D^k f(z)}{z} = \frac{f(z)}{z} \left( \sum_{m=0}^{k-1} \frac{D^m f(z)}{z} \right), \quad 0 \leq k \leq n+1,
\]

which yields

\[
|\arg \frac{D^k f(z)}{z} | = |\arg \frac{f(z)}{z} | + \sum_{m=0}^{k-1} |\arg \frac{D^m f(z)}{z} | . \tag{3.1}
\]

Since \( R_{n+1}(\alpha) \subset R_n(\alpha) \ \forall n \in \mathbb{N}_0 \), it follows that \( f \in R_m(\alpha) \) for each \( m(0 \leq m \leq n) \).

Setting

\[
\frac{D^{m+1} f(z)}{D^m f(z)} = q_m(z), \quad (0 \leq m \leq n), \tag{3.2}
\]

we note that \( \text{Re}(q_m(z)) \geq (m+\alpha)/(m+1) \).

Therefore, the function

\[
\omega(z) = \frac{(q_m(z) - \frac{m+\alpha}{m+1}) - (1 - \frac{m+\alpha}{m+1})}{(q_m(z) - \frac{m+\alpha}{m+1}) + (1 - \frac{m+\alpha}{m+1})}
\]

is analytic with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \Delta \). Hence by Schwarz's Lemma,
\[ \left| \frac{q_m(z) - 1}{q_m(z) + 1 - 2(m+a)/(m+1)} \right| < |z| \]

for \( z \in \Delta \). Now it is easy to see that the values of \( q_m(z) \) are contained in the circle of Appolonius whose centre is at the point
\[ (m+1-(m+2a-1)r^2)/((1+m)(1-r^2)) \]
and has radius \( 2(1-a)r/((m+1)(1-r^2)) \).

Thus \( \max |\arg q_m(z)| \) is attained at the points where \( z \in \Delta \)

\[ \arg q_m(z) = \pm \sin^{-1}\left(\frac{2(1-a)r}{m+1-(m+2a-1)r}\right) \]

which gives

\[ |\arg \frac{D^{m+1}f(z)}{D^m f(z)}| \leq \sin^{-1}\left(\frac{2(1-a)r}{m+1-(m+2a-1)r}\right), \quad (3.3) \]

for \( 0 \leq m \leq n \), and \( z \in \Delta \).

Next, note that \( R_n(\alpha) : S^\alpha(\alpha), n \geq 0 \), and \( f \in S^\alpha(\alpha) \) if and only if \( F(z) = \int (f(z)/z)dz \) is in \( K(\alpha) \). But for \( F \in K(\alpha) \), we have

\[ |\arg F'(z)| \leq 2(1-\alpha)\sin^{-1}r \quad (|z| = r) \]

Thus \( f \in R_n(\alpha) \) implies

\[ |\arg \frac{f(z)}{z}| \leq 2(1-\alpha)\sin^{-1}r \quad (3.4) \]

Applying (3.3) and (3.4) to (3.1) we obtain the result.

For \( n = 0 \), we obtain

COROLLARY If \( f \in S^\alpha(\alpha) \), then (3.4)

and

\[ |\arg f'(z)| \leq 2(1-\alpha)\sin^{-1}r + \sin^{-1}\left(\frac{2(1-a)r}{1-(2a-1)r^2}\right) \]

REMARK The case \( n = 0, \alpha = 0 \) way proved by Krzyz [10].

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