NORM-PRESERVING L-L INTEGRAL TRANSFORMATIONS

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ABSTRACT. In this paper we consider an L-L integral transformation $G$ of the form $F(x) = \int_0^x G(x,y)f(y)dy$, where $G(x,y)$ is defined on $D = \{(x,y): x \geq 0, y \geq 0\}$ and $f(y)$ is defined on $[0,\infty)$. The following results are proved: For an L-L integral transformation $G$ to be norm-preserving, $\int_0^x |G(x,t)|dx = 1$ for almost all $t \geq 0$ is only a necessary condition, where $G(x,t) = \lim_{h \to 0} \inf_{t \leq h} \int_t^{t+h} G(x,y)dy$ for each $x \geq 0$. For certain $G$'s, $\int_0^x |G(x,t)|dx = 1$ for almost all $t \geq 0$ is a necessary and sufficient condition for preserving the norm of certain $f \in L$. In this paper the analogous result for sum-preserving L-L integral transformation $G$ is proved.

KEY WORDS AND PHRASES. $\ell$-$\ell$ method. L-L integral transformation. Absolutely continuity of integrals. Fubini-Tonelli Theorem.

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1. INTRODUCTION.

The well-known summability method defined by a $\ell$-$\ell$ matrix $A = (a_{nk})$, mapping from $\ell$ into $\ell$, is sum-preserving if and only if for each $k$, $\sum_{n \geq 1} a_{nk} = 1$. In our present study we also discuss conditions under which $G$ defined by $G(x,y)$, mappings $L$ into $L$, is norm-preserving or sum-preserving.

2. NOTATION.

The notation and terms used are:

The statement that $f$ is Lebesgue integrable on $[0,\infty)$ means that for every $u > 0$, if $f$ is Lebesgue integrable on $[0,u]$ and that $\int_0^u f(x)dx$ tends to a finite limit as $u \to \infty$.

$L$ - the space of functions that are Lebesgue integrable on $[0,\infty)$ with norm $\|f\| = \int_0^\infty |f(x)|dx$.

$D$ - the first quadrant of the plane, i.e., $D = \{(x,y): x \geq 0, y \geq 0\}$.

$G$ - an integral transformation, $G: f \mapsto F$, of the form (*) $F(x) = \int_0^x G(x,y)f(y)dy$, for all $x \geq 0$, where $f$ is defined on $[0,\infty)$ and $G(x,y)$ defined on $D$.

$G_L$ - the subcollection of $G$ such that $F \in L$ whenever $f \in L$.

$L_\infty$ - the space of functions which are measurable and essentially bounded on $[0,\infty)$ with norm $\|f\|_\infty = \text{ess sup}_{x \geq 0} |f(x)|$. 
3. MAIN THEOREM

THEOREM 1. If \( G \in GL \) and for every \( f \in L \)

\[ \int_0^\infty |F(x)|dx = \int_0^\infty |f(y)|dy, \]

then for almost all \( y \geq 0 \),

\[ \int_0^\infty |G_*(x,y)|dx = 1, \]

where \( G_*(x,y) = \lim \inf_{h \to 0} \frac{1}{h} \int_y^{y+h} G(x,t)dt \), for each \( x \geq 0 \).

PROOF. Suppose that there is a set \( A = \{ y \neq 0 \} \) satisfying \( 0 < m_A < \infty \) such that either \( \int_0^\infty |G_*(x,y)|dx > 1 \) for all \( y \in A \) or \( \int_0^\infty |G_*(x,y)|dx < 1 \) for all \( y \in A \).

Since \( G \in GL \), for each \( x \geq 0 \), it follows from Theorem (T.S.T.), see [1], that for every measurable set \( A \) of finite measure, \( \int_A G(x,y)dy < \infty \), without loss of generality, we can assume that \( A \) is a bounded measurable set.

Case i). Suppose that for all \( y \in A \), \( \int_0^\infty |G_*(x,y)|dx < 1 \). Without loss of generality we assume that for each \( y \in A \)

\[ \int_0^\infty |G_*(x,y)|dx < 1 - \varepsilon, \]

where \( \varepsilon \) is a small positive number. Let \( f(y) = \chi_A(y) \) then

\[ F(x) = \int_0^\infty G(x,y) \chi_A(y)dy = \int_A G(x,y)dy, \]

and

\[ \|F\| = \int_0^\infty \int_0^\infty G(x,y) \chi_A(y)dy|dx \]
\[ \leq \int_0^\infty \int_A |G(x,y)|dydx. \]

Since for each \( x \geq 0 \), \( G_*(x,y) = G(x,y) \) for almost all \( y \geq 0 \), see [2, Theorem 5. P. 255], so it follows from the Fubini-Tonelli Theorem that

\[ \|F\| \leq \int_0^\infty \int_A |G(x,y)|dydx \]
\[ = \int_0^\infty \int_A |G_*(x,y)|dydx \]
\[ = \int_A \int_0^\infty |G_*(x,y)|dydx \]
\[ \leq \int_A (1 - \varepsilon)dy \]
\[ = (1 - \varepsilon) m_A \]
\[ < m_A = \| \chi_A \|. \]

Hence, for case i), \( G \) is not norm-preserving.

Case ii). Suppose that for all \( y \in A \), \( \int_0^\infty |G_*(x,y)|dx > 1 \). Without loss of generality, we assume that for each \( y \in A \)

\[ \int_0^\infty |G_*(x,y)|dx > 1 + \varepsilon. \]

where \( \varepsilon \) is a small positive number. Let \( f(y) = \chi_A(y) \); then \( F(x) = \int_A G(x,y)dy \)

for all \( x \in [0,\infty) \). If \( F(x) = 0 \) for almost all \( x \in [0,\infty) \), then

\[ \|F\| = 0 < m_A = \| \chi_A \| = \| f \|, \]

and we're done.

Suppose that \( F(x) \neq 0 \) for all \( x \) in some set with positive measure. Since \( G \in GL \), so it follows from Theorem (T. S. T) by author, see [1], \( G_*(x,y) \) is measur-
able on $D$ and $\int_0^\infty |G_*(x,y)| \, dy < M$ for almost all $y \geq 0$, where $M$ is a constant. Thus
\[
\int_A G_*(x,y) \, dy = \int_A G(x,y) \, dy \quad \text{and} \quad \int_A \int_0^\infty |G_*(x,y)| \, dx \, dy < M \cdot A < \infty , \quad \text{and}
\int_0^\infty \int_A |G_*(x,y)| \, dy \, dx = \int_A \int_0^\infty |G_*(x,y)| \, dx \, dy < \infty .
\]

Given $\varepsilon > \eta > 0$, there is an $X_0 > 0$ such that
\[
\int_{X_0}^\infty \int_A |G_*(x,y)| \, dx \, dy < \eta \cdot mA/2 < \varepsilon \cdot mA/4 .
\]

It follows that there is at least a subset $A_0 \subseteq A$, having positive measure and for all $y \in A_0$, satisfying
\[
\int_{X_0}^\infty |G_*(x,y)| \, dx < \varepsilon/8
\]
and from \(\int_0^\infty |G_*(x,y)| \, dx > 1 + \varepsilon \) for each $y \in A$ that
\[
\int_0^\infty |G_*(x,y)| \, dx > 1 + 3\varepsilon/4
\]
for all $y \in A_0$. Let $E = \{(x,y) \in [0,X_0] \times A_0 : |G_*(x,y)| < \varepsilon/4X_0 \}$ and for any $y \in A_0$, let $E_y = \{(x_1 \in [0,X_0] : (x_1,y) \in E \}$. Then $0 \leq mE_y \leq X_0$ for all $y \in A_0$. Since
\[
\int_{A_0} \int_0^\infty |G_*(x,y)| \, dx \, dy \leq \int_{A_0} \int_0^\infty |G_*(x,y)| \, dx \, dy
\]
so it follows from the absolute continuity of the integral that there is a $\delta > 0$ such that for every measurable set $H \subseteq [0,X_0] \times A_0$ satisfying $mH < \delta$, and
\[
\int_H |G_*(x,y)| \, dx \, dy < \eta \cdot mA_0/4 .
\]

If $\int_{A_0} G(x,y) \, dy = 0$ for all $x \geq 0$, then
\[
\|F\| = \int_0^\infty \int_{A_0} G(x,y) \chi_{A_0} \, dy \, dx
\]
\[
= \int_0^\infty \int_{A_0} G(x,y) \, dy \, dx
\]
\[
= 0 < mA_0 = \|F\| ,
\]
and we're done. So we suppose that $\int_{A_0} G(x,y) \, dy \neq 0$ for some set of $x \geq 0$ with positive measure. By the Generalization of Luzin's Theorem we can choose a closed set $F \subseteq [0,X_0] \times A_0$ such that if $H = [0,X_0] \times A_0 \setminus F$ then $mH < \delta$ and
\[
\int_H |G_*(x,y)| \, dx \, dy < \eta \cdot mA_0/4 ,
\]
and $G_*(x,y)$ is continuous over $F$. It is clear that $G_*(x,y)$ is uniformly continuous on $F$. Thus we can have a finite number $N$ of subsets $A_1$ of $mA_1 > 0$ of set $A_0$ such that $F = \bigcup_{i=1}^N [0,X_0] \times A_1$ and within each strip $[0,X_0] \times A_1$ for each $x \in [0,X_0]$ the value of $G_*(x,y)$ are close to one another. More precisely, for $(x,y')$, $(x,y'') \in [0,X_0] \times A_1$ and $(x,y')$, $(x,y'') \notin E$, $|G_*(x,y') - G_*(x,y'')| < \varepsilon/4X_0$. Then for each $A_1$ there are three sets $A_1^+$, $A_1^-$ and $E_y$ of $x \in [0,X_0]$, such that
\[
G_*(x,y) > 0, \quad \text{if} \quad (x,y) \in F \cap (A_1^+ \times A_1)
\]
\[
G_*(x,y) < 0, \quad \text{if} \quad (x,y) \in F \cap (A_1^- \times A_1)
\]
and
\[
|G_*(x,y)| < \varepsilon/4X_0, \quad \text{if} \quad (x,y) \in E_y \times A_1 .
\]
Hence, if \((x,y) \in F \cap [0,X_0] \times A_1\), then
\[
\int_0^{X_0} \int_0^y G(x,y) \, dx \, dy = \int_0^{X_0} \int_{A_1} G(x,y) \, dx \, dy
\]
\[
= \int_0^{X_0} \int_{A_1} G_*(x,y) \, dx \, dy
\]
\[
= \int_{E_y} \int_{A_1} G_*(x,y) \, dx \, dy + \int_{E_x} \int_{A_1} G_*(x,y) \, dx \, dy
\]
\[
= \int_{A_1} \int_{E} G_*(x,y) \, dx \, dy + \int_{A_1} \int_{E} (-G_*(x,y)) \, dx \, dy
\]
\[
= \int_{A_1} \int_{E} G_*(x,y) \, dx \, dy + \int_{A_1} \int_{E} (-G_*(x,y)) \, dx \, dy
\]
\[
> (1 + 3e/4)m_{A_1} - \epsilon \cdot m_{A_1}/8 \quad \text{(since } m_{E_x} < X_0)\,.
\]

If \(m[H \cap [0,X_0] \times A_1] = 0\) for some \(A_1 \in \{A_1\}_1^N\), then for such an \(A_1\),
\[
\|F\| = \int_0^{X_0} \int_0^y G(x,y) \, dx \, dy
\]
\[
= \int_0^{X_0} \int_{A_1} G(x,y) \, dx \, dy
\]
\[
= \int_0^{X_0} \int_{A_1} G_*(x,y) \, dx \, dy + \int_0^{X_0} \int_{A_1} G_*(x,y) \, dx \, dy
\]
\[
\geq \int_0^{X_0} \int_{A_1} G_*(x,y) \, dx \, dy - \int_0^{X_0} \int_{A_1} G_*(x,y) \, dx \, dy
\]
\[
= \int_0^{X_0} \int_{A_1} G_*(x,y) \, dx \, dy
\]
\[
> (1 + 3e/4)m_{A_1} - \epsilon m_{A_1}/8
\]
\[
m_{A_1} = \|x_{A_1}\|, \text{ and we're done}.
\]

If \(m[H \cap [0,X_0] \times A_1] \neq 0\) for all \(A_1 \in \{A_1\}_1^N\), then there is at least an \(A_1\) such that
\[
\int_{H \cap [0,X_0] \times A_1} |G_*(x,y)| \, dx \, dy < (n \cdot m_{A_0}/4) \cdot m_{A_1}/m_{A_0},
\]
and for such an $A_1$, 
\[
\|F\| = \int_0^\infty \left| \int_0^\infty G(x,y)\chi_{A_1}(y)dy \right| dx
\]
\[
= \int_0^\infty \left( \int_0^\infty G_\ast(x,y)\chi_{A_1}(y)dy \right) dx + \int_0^\infty \left( \int_0^\infty G_\ast(x,y)\chi_{A_1}(y)dy \right) dx
\]
\[
\geq \int_0^\infty \left| \int_0^\infty G_\ast(x,y)dy \right| dx - \int_0^\infty \left| \int_0^\infty G_\ast(x,y)dy \right| dx
\]
\[
(x,y) \in F
\]
\[
\geq \int_0^\infty G_\ast(x,y)dy - n \cdot mA_1/4
\]
\[
(x,y) \in F
\]
\[
> (1 + 3\epsilon/4) mA_1 - 2 \epsilon mA_1/8
\]
\[
= (1 + \epsilon/2) mA_1
\]
\[
> mA_1 = \|\chi_{A_1}\| .
\]
Hence, case (ii) we have proved that $G$ is not norm-preserving and so the proof is complete.

Theorem 1 shows us that if $G \in \mathcal{G}_L$, for almost all $y > 0$, \( \int_0^\infty |G_\ast(x,y)| dx = 1 \) is a necessary condition for \( \int_0^\infty |F(x)| dx = \int_0^\infty |f(y)| dy \) whenever $f \in L$. The next example will tell us that for almost all $y > 0$, \( \int_0^\infty |G_\ast(x,y)| dx = 1 \) is not a sufficient condition for \( \int_0^\infty |F(x)| dx = \int_0^\infty |f(y)| dy \) for every $f \in L$.

But the following theorem will show that for certain $G$'s, \( \int_0^\infty |G_\ast(x,y)| dx = 1 \) is a necessary and sufficient condition for preserving the norms of certain $f \in L$.

**Example.** Define
\[
G(x,y) = \begin{cases} 
-1/4x^{1/2} & \text{if } x \in (0,1), \\
1/2x^2 & \text{if } x \in [1,\infty),
\end{cases}
\]
for all $y > 0$; 

and 
\[
f(y) = \begin{cases} 
-2/(y + 1)^2 & \text{if } y \in (0,1), \\
10/(y + 1)^2 & \text{if } y \in [1,\infty).
\end{cases}
\]
Then 
\[
\int_0^\infty |f(y)| dy = \int_0^1 2/(y + 1)^2 dy + \int_1^\infty 10/(y + 1)^2 dy
\]
\[
= -2(y + 1)^{-1}\bigg|_0^1 - 10(y + 1)^{-1}\bigg|_1^\infty
\]
\[
= -1 + 2 + 5 = 6,
\]
and 
\[
\int_0^\infty |G_\ast(x,y)| dx = \int_0^1 1/4x^{1/2} dx + \int_1^\infty 1/2x^2 dx
\]
\[
= 2x^{1/2}\bigg|_0^1 - 1/2x^{1/2}\bigg|_1^\infty
\]
\[
= 1/2 + 1/2 = 1.
\]
But 
\[
F(x) = \int_0^\infty G(x,y)f(y)dy
\]
\[
= \begin{cases} 
\int_0^\infty (-1/4x^{1/2}) f(y)dy & \text{if } x \in (0,1), \\
\int_0^\infty (1/2x^2) f(y)dy & \text{if } x \in [1,\infty)
\end{cases}
\]
where
\[- \int_0^{\infty} (1/4x^{1/2}) f(y) dy = -1/4x^{1/2} \left[ \int_0^1 -2/(y + 1)^2 dy + \int_1^{\infty} 10/(y + 1)^2 dy \right] = -1/4x^{1/2} \left[ -2(-1)(y + 1)^{-1} \right]_0^1 + 10(-1)(y + 1)^{-1} \right]_1^{\infty} = -1/4x^{1/2} [1 - 2 + 5] = -1/2x^{1/2}, \text{ if } x \in (0,1), \]

and
\[\int_0^{\infty} (1/2x^2) f(y) dy = (1/2x^2) \left[ \int_0^1 -2/(y + 1)^2 dy + \int_1^{\infty} 10/(y + 1)^2 dy \right] = (1/2x^2)(1 - 2 + 5) = 2/x^2, \text{ if } x \in [1,\infty). \]

Therefore
\[\int_0^{\infty} |F(x)| dx = \int_0^1 1/x^{1/2} dx + \int_1^{\infty} 1/x^2 dx = 2x^{1/2} \left[ \right]_0^1 + 2(-1)x^{-1} \left[ \right]_1^{\infty} = 2 + 2 = 4 \neq 6 = \int_0^{\infty} |f(y)| dy. \]

**THEOREM 2.** Suppose that \( G(x,y) \) is a nonnegative function on \( D \) and \( G \in GL; \) then the following are equivalent;

i) \( F = \| f \| \) whenever \( f \in L \) and \( f(y) \geq 0 \) on \( [0,\infty) \);

ii) \( F = \| f \| \) whenever \( f \in L \) and \( f(y) \leq 0 \) on \( [0,\infty) \);

iii) \( F = \| f \| \) whenever \( f \in L \), if \( F(x) = \int_0^{\infty} G(x,y) |f(y)| dy \);

iv) \( \int_0^{\infty} G_s(x,y) dx = 1, \) for almost all \( y \geq 0. \)

**PROOF.** Since \( \| f \| = \int_0^{\infty} |f(y)| dy, \) \( F(x) = \int_0^{\infty} G(x,y) f(y) dy \) and
\[\| F \| = \int_0^{\infty} |F(x)| dx, \]

it is clear that i) is equivalent to i). We now prove that i) is equivalent to iv). Assuming that \( G(x,y) \geq 0 \) on \( D \) and \( f(y) \geq 0 \) for all \( y \in [0,\infty), \) we have \( f(y)G(x,y) \geq 0 \) on \( D. \) Hence
\[F(x) = \int_0^{\infty} G(x,y) f(y) dy \geq 0, \]

so
\[\| F \| = \int_0^{\infty} |F(x)| dx = \int_0^{\infty} F(x) dx, \]

and
\[\| F \| = \int_0^{\infty} \int_0^{\infty} G(x,y) f(y) dy dx. \]

By the Fubini-Tonelli Theorem and for each \( x \geq 0, \) \( G_s(x,y) = G(x,y) \) for almost all \( y \geq 0, \)
\[\| F \| = \int_0^{\infty} |f(y)| \int_0^{\infty} G_s(x,y) dx dy. \]

Hence,
\[\| F \| = \| f \| \text{ if and only if } \int_0^{\infty} G_s(x,y) dx = 1 \text{ for almost all } y \geq 0. \]
Next we prove that iii) is equivalent to iv). Let

\[ f^+ = \begin{cases} 
  f(y), & \text{if } f(y) \geq 0 \\
  0, & \text{if } f(y) < 0 \\
  f(y), & \text{if } f(y) < 0 \\
  0, & \text{if } f(y) > 0 
\end{cases} \]

Since \( G(x,y) \geq 0 \) on \( D \), so whenever \( f \in L \)

\[ F^+ = \int_0^\infty G(x,y)f^+(y)dy \geq 0, \text{ for all } x \geq 0; \]

\[ F^- = \int_0^\infty G(x,y)f^-(y)dy \geq 0, \text{ for all } x \geq 0; \]

and

\[ f(y) = f^+ - f^- \]

if \( F(x) = \int_0^\infty G(x,y)|f(y)|dy \), then

\[ |F(x)| = F^+ + F^- \]

It follows from i) that

\[ \|F(x)\| = \int_0^\infty |F(x)|dx = \int_0^\infty |f(y)|dy \]

if and only if

\[ \int_0^\infty G^*(x,y)dx = 1 \text{ for almost all } y \geq 0. \]

We are also interested in the analogous sum-preserving question for L-L integral transformations, viz., when is \( \int_0^\infty F(x)dx = \int_0^\infty f(y)dy \) whenever \( f \in L \)?

Next we give the definition of sum-preserving for L-L integral transformations and a result concerning it.

DEFINITION. The integral transformation \( G \in GL \) is said to be sum-preserving if and only if

\[ \int_0^\infty F(x)dx = \int_0^\infty f(y)dy \]

for all \( f(y) \in L \), where \( F(x) = \int_0^\infty G(x,y)f(y)dy \).

COROLLARY. Suppose that \( G(x,y) \) is a nonnegative function on \( D \) and \( G \in GL; \) then \( G \) is a sum-preserving transformation whenever \( f \in L \) if and only if \( \int_0^\infty G^*(x,y)dx = 1 \) for almost all \( y \geq 0 \).

PROOF. Since \( f \in L, f = f^+ - f^- \), where

\[ f^+ = \begin{cases} 
  f(y), & \text{if } f(y) \geq 0 \\
  0, & \text{if } f(y) < 0 \\
  f(y), & \text{if } f(y) < 0 \\
  0, & \text{if } f(y) > 0 
\end{cases} \]

\[ f^- = \begin{cases} 
  -f(y), & \text{if } f(y) < 0 \\
  0, & \text{if } f(y) \geq 0 
\end{cases} \]

and

\[ \int_0^\infty f(y)dy = \int_0^\infty f^+dy - \int_0^\infty f^-dy \].
Then
\[ F(x) = \int_0^\infty G(x,y)f(y)dy \]
\[ = \int_0^\infty G(x,y)[f^+ - f^-]dy \]
\[ = \int_0^\infty G(x,y)f^+(y)dy - \int_0^\infty G(x,y)f^-(y)dy \]
and
\[ \int_0^\infty F(x)dx = \int_0^\infty \int_0^\infty G(x,y)f^+(y)dydx - \int_0^\infty \int_0^\infty G(x,y)f^-(y)dydx \]
\[ = \int_0^\infty \int_0^\infty G(x,y)f^+(y)dydx - \int_0^\infty \int_0^\infty G(x,y)f^-(y)dydx . \]

By the Fubini-Tonelli Theorem
\[ \int_0^\infty \int_0^\infty G(x,y)f^+(y)dydx = \int_0^\infty f^+(y) \int_0^\infty G(x,y)dx dy ; \]
and
\[ \int_0^\infty \int_0^\infty G(x,y)f^-(y)dydx = \int_0^\infty f^-(y) \int_0^\infty G(x,y)dx dy . \]

Thus
\[ \int_0^\infty \int_0^\infty G(x,y) f^+(y)dydx = \int_0^\infty f^+(y)dy ; \]
and
\[ \int_0^\infty \int_0^\infty G(x,y) f^-(y)dydx = \int_0^\infty f^-(y)dy \]
if and only if
\[ \int_0^\infty G(x,y)dx = 1 \text{ for almost all } y \geq 0 . \]

Therefore
\[ \int_0^\infty F(x)dx = \int_0^\infty f^+dy - \int_0^\infty f^-dy \]
if and only if
\[ \int_0^\infty G(x,y)dx = 1 \text{ for almost all } y \geq 0 , \]
i.e.,
\[ \int_0^\infty F(x)dx = \int_0^\infty f(y)dy \]
if and only if
\[ \int_0^\infty G(x,y)dx = 1 \text{ for almost all } y \geq 0 . \]

The proof is completed.

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