SEPARATION AXIOMS FOR
PARTIALLY ORDERED CONVERGENCE SPACES

REINO VAINIO

Department of Mathematics
Abo Akademi
SF-205000 Abo 50, Finland

(Received October 24, 1984)

ABSTRACT. In partially ordered convergence spaces, separation axioms are introduced
and then related to the concept of complete separatedness due to Nachbin as well as
to connectedness concepts. A method to generate new separation axioms is studied.

KEY WORDS AND PHRASES. Convergence space, partial order, separation axioms,
connectivity, interval topology.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. Primary 54A20, Secondary 54F05.

1. INTRODUCTION

Lately, convergence structures more general than topologies have proved to be
effective tools in posets and lattices (cf. Kent [1], Erné and Weck [2], and Ball
[3]). In this note we shall study some relations of interdependence between a con-
vergence structure and a partial order, hereby concentrating upon separation axioms
and related matters. This is done within the realms of partially ordered (po) conver-
gence spaces. In the po topological case, most of the material in Section 3 is known
from Nachbin [4] and Mc Cartan [5]. The interplay between separation axioms and
connectivity properties, as worked out in Sections 4 and 5, has not been studied in
po topological spaces. For a correspondence in topological spaces without order,
reference is made to Preuss [6, Ch. 6].

Here convergence structure is used in the sense of Kent [7]; the precise defini-
tion is stated in Section 2. A partially ordered (po) convergence space is a triplet
\((X, \mathcal{Q}, \leq)\), where \(X\) is a set, \(\mathcal{Q}\) a convergence structure on \(X\) and \(\leq\) a partial order
relation on \(X\). Obviously, this is a generalization of the partially ordered (po)
topological spaces introduced by Nachbin. We can regard every convergence space as a
po convergence space, where the order in question is discrete. Every definition
which we propose for po convergence spaces shall be subject to the following criteria:
(i) For po topological spaces, it reduces to the classic definition in the sense of
Nachbin.
(ii) For discrete order, it coincides with a natural definition in convergence space
theory.
Hence it is clear that every definition for po convergence spaces defines a natural compatibility between convergence structure and partial order.

There are natural, well-known non-topological po convergence structures, for instance order convergence on posets (cf. Kent [1], Erné and Weck [2]) and several structures defined by R.N. Ball. Before proceeding, we wish to mention the expository article Choe [8], which covers the main streams of research in po topological spaces up to recent date.

2. PRELIMINARIES

For later use, we gather a few definitions and notations concerning convergence and order. For our aim, the following basic definition is the proper one.

DEFINITION 2.1. (Kent [7]). Let \( X \) be a set. A convergence structure \( q \) on \( X \) is a map \( q \), which assigns to every \( x \in X \) a set \( q(x) \) of filters on \( X \) being subject to the conditions below (\( x \in X \)):

1. \( [x] \in q(x) \)
2. \( \mathcal{F} \in q(x) \text{ and } \mathcal{G} \supseteq \mathcal{F} = \mathcal{G} \in q(x) \)
3. \( \mathcal{F} \in q(x) = \mathcal{F} \cap [x] \in q(x). \)

Hereby, \([x]\) denotes the ultrafilter generated by \( \{x\} \). The pair \((X,q)\) is called a convergence space.

Obviously, this definition provides a generalization of topological structure and topological space. We do not require the filters in \( q(x) \) to form full intersection ideals, since we wish to consider order convergence on posets as a special case of the convergence structures being treated here. For theory and application of convergence structures, we recommend the book Gähler [9].

If \( q \) is a convergence structure on the set \( X \), then \( tq \) is the finest topology on \( X \) being coarser than \( q \). The notion of open (closed) set in a space \((X,q)\) always refers to the topological space \((X,tq)\). Let \((X,q)\) be a convergence space and take \( A \subseteq X \). Then, \( \overset{q}{}{A} \) denotes the set of all \( x \in X \) for which \( q(x) \) contains some \( \mathcal{F} \) with \( A \in \mathcal{F} \). For \((X,q)\) and \((Y,r)\) given convergence spaces, a continuous map \( f : (X,q) \rightarrow (Y,r) \) is a map \( f : X \rightarrow Y \) for which \( x \in X, \mathcal{F} \in q(x) = f(\mathcal{F}) \in r(f(x)) \).

Occasionally, a po convergence space \((X,q,\leq)\) shall be denoted by \( X \) only, and then, instead of \( \mathcal{F} \in q(x) \) shall be written \( \mathcal{F} \rightarrow x \). The topological modification \( tX \) of a given po convergence space \( X \) is the po topological space \((X,tq,\leq)\). If \( X \) and \( Y \) are po convergence spaces, a morphism \( \phi : X \rightarrow Y \) is an increasing continuous map \( \phi \) from \( X \) to \( Y \).

In a given poset, \( x \nleq y \) denotes that \( x \leq y \) is false, and \( x \nleq y \) is equivalent to \( x \nleq y \) and \( y \nleq x \). If \( \mathcal{F} \) is a set, then \( i(\mathcal{F}) (d(\mathcal{F})) \) denotes the smallest increasing (decreasing) set containing \( \mathcal{F} \), and \( \mathcal{F}^{*} (\mathcal{F}^{-}) \) denotes the set of all upper (lower) bounds of \( \mathcal{F} \). Instead of \( \{a\}^{*} (\{a\}^{-}) \) is written \( a^{*} (a^{-}) \). On a given poset, the interval topology is the coarsest topology for which all rays, i.e., the sets of the form \( a^{*} \) or \( a^{-} \), are closed sets.

3. SEPARATION AXIOMS

For definitions and results in the case of po topological spaces, reference is made to Nachbin [4], Mc Cartan [5] and Ward [10]. Synonymously with \( T_{1} \)-ordered po topological space, however, the concepts of semi-closed partial order and semi-
continuous partial order are used in Nachbin [4] and Ward [10], respectively. $T_1$-ordered and $T_2$-ordered po convergence spaces were introduced in Kent and Richardson [11].

**DEFINITION 3.1.** A po convergence space $X$ is lower (upper) $T_1$-ordered, if for every pair $a \nless b$ in $X$ and for every $F \triangleleft a$ ($F \triangleright a$) there is a set $F \in \mathcal{F}$ such that $x \nless b$ ($a \nless x$) for all $x \in F$. This separation axiom is denoted by ord $T_{1L}$ (ord $T_{1U}$).

**THEOREM 3.2.** Let $X$ be a po convergence space. The following conditions are equivalent:

(i) $X$ is ord $T_{1L}$.

(ii) For every pair $a \nless b$ in $X$, for every $F \triangleleft a$ and for every $F \in \mathcal{F}$, $b \nless F^*$.

(iii) For every pair $a \nless b$ in $X$, and for every $F \triangleleft a$ there is an increasing set $V \in \mathcal{F}$ with $b \nless V$.

(iv) For every $a \in X$ the ray $a^+$ is a closed set.

Corresponding characterizations hold true for ord $T_{1U}$ (with obvious changes only).

**PROOF.** (i) $\Rightarrow$ (ii). According to (i), the filter $F \triangleleft a$ in (ii) contains some $F_0$ with $x \nless b$ for all $x \in F_0$. Since every $F \in \mathcal{F}$ intersects $F_0$, we are through. (ii) $\Rightarrow$ (iii). Let $X$ satisfy (ii), take a $\nless b$ in $X$ and $F \triangleleft a$, then write

$$\mathcal{F} = \bigcap_{k \in K} G_k$$

where $G_k$ ($k \in K$) are the ultrafilters finer than $F$. Now, fix $k \in K$. According to (ii), every set $G_{kj}$ in $G_k = (G_{kj})_{j \in J}$ contains an element $s_j$ for which $s_j \nless b$. Denote $S_k = \{s_j | j \in J\}$. Since $S_k$ intersects all sets in the ultrafilter $G_k$, it follows $S_k \in G_k$. Thus $V_k = i(S_k) \in G_k$, $b \nless V_k$, and the set

$$V = \bigcup_{k \in K} V_k$$

in an increasing set in $F$ with $b \nless V$.

(iii) $\Rightarrow$ (iv). Assume (iii), take $a \in X$ and $x \nless a^+$, i.e. $x \nless a$. For any $F \triangleleft x$ there is an increasing set $V \in \mathcal{F}$ with $b \nless V$. Thus $V \cap a^+ = \emptyset$, and $X \backslash a^+$ is an open set.

(iv) $\Rightarrow$ (i). If $X$ satisfies (iv), then the topological modification $tX$ is ord $T_{1L}$ (Mc Cartan [5]), and (i) follows.

**COROLLARY 3.3.** A po convergence space $X$ is ord $T_{1L}$ (ord $T_{1U}$), if and only if the topological modification $tX$ is.

**DEFINITION 3.4.** A po convergence space is $T_1$-ordered, if it is both $T_{1L}$-ordered and $T_{1U}$-ordered. This separation axiom is denoted by ord $T_1$.

**THEOREM 3.5.** A po convergence space $(X,q,\leq)$ is ord $T_1$, if and only if the convergence structure $q$ is finer than the interval topology of the po relation $\leq$.

Hence, in po convergence spaces satisfying ord $T_1$, all maximal chains are closed sets.

**REMARK 3.6.** In the Introduction we stated two criteria, (i) and (ii), which new definitions in the theory of po convergence spaces should meet. The definition of ord $T_{1L}$ (ord $T_{1U}$) fills (i). In case of discrete order, both ord $T_{1L}$ and ord $T_{1U}$
coincide with the separation axiom $T_1$ for convergence spaces. Thus also (ii) is satisfied. Naturally, the definition of $ord T_1$ also satisfies both criteria.

Finally we note that if the po convergence space $(X,q,\leq)$ is $ord T_{1L}$ or $ord T_{1U}$, then the convergence space $(X,q)$ is $T_0$. Moreover, if $(X,q,\leq)$ is $ord T_1$, then $(X,q)$ is $T_1$.

DEFINITION 3.7. A po convergence space $X$ is $T_2$-ordered, if for every pair $a \not\leq b$ in $X$, and for every $F \to a$ and $G \to b$, there are sets $F \in F$ and $G \in G$ such that $x \not\leq y$ for all $x \in F$, $y \in G$. This separation axiom is denoted by $ord T_2$.

THEOREM 3.8. (Kent and Richardson [11]). Let $X$ be a po convergence space. The following conditions are equivalent:

(i) $X$ is $ord T_2$.
(ii) For every pair $a \not\leq b$ in $X$, for every $F \to a$ and $G \to b$, there is an increasing set $F \in F$ and a decreasing set $G \in G$ for which $F \cap G = \emptyset$.
(iii) The graph of the partial order of $X$ is a closed set in the product convergence space $X \times X$.

REMARK 3.9. If the order relation of a po convergence space is a total order, then $ord T_1 = ord T_2$. This follows from the corresponding statement in the po topological case (cf. Ward [10]) and from Corollary 3.3. There is a vast literature on totally ordered topological spaces satisfying $ord T_1$, for instance within the realms of orderability theory (cf. Eilenberg [12]). In this paper, the special case of totally ordered convergence spaces is not treated.

REMARK 3.10. For po topological spaces, the definition of $ord T_2$ coincides with the classic definition of closed order (Nachbin [4], Mc Cartan [5]). If the order of a po convergence space is discrete, then $ord T_2$ coincides with the classic separation axiom $T_2$ for convergence spaces. (Thus, Theorem 3.8 can be regarded as a generalization of the usual characterization "A convergence space is $T_2$, if and only if the diagonal is a closed set in the product space").

Moreover, if $(X,q,\leq)$ is $ord T_2$, then the convergence space $(X,q)$ is $T_2$. Every $T_2$-ordered po convergence space is also $T_1$-ordered. It is possible for $(X,q,\leq)$ to be $ord T_2$, without the topological modification $(X,tq,\leq)$ having that property.

Below, we propose two variants of regularity for po convergence spaces. For the main part, the case of lower regularity ($ord T_{3L}$) is treated.

DEFINITION 3.11. A po convergence space $(X,q,\leq)$ is lower $T_3$-ordered, if for every closed decreasing set $M$, for every $x \not\leq M$ and for every $G \in q(x)$, there is a set $G \in G$ for which $M \cap \overline{G} = \emptyset$. This separation axiom is denoted by $ord T_{3L}$.

DEFINITION 3.12. A po convergence space $(X,q,\leq)$ is strongly lower $T_3$-ordered, if for every $x \in X$ and every $F \in q(x)$, there is a filter $G \in q(x)$ for which $\overline{F} \supseteq \overline{G}$. This separation axiom is denoted by $st ord T_{3L}$.

By $i(F)$ is meant the filter on $X$, which is generated by the sets $i(F)$, $F \in F$. It is easily verified that $st ord T_{3L} = ord T_{3L}$. The reverse implication is false, even if $q$ is a topology. Then $ord T_{3L}$ coincides with the classic
definition of lower regularity (Mc Cartan [5]), while \( \text{st ord } T_{3L} \) becomes "For every \( x \in X \) and for every increasing neighborhood \( V \) of \( x \), there is an increasing neighborhood \( W \) of \( x \), for which \( \overline{W} \subseteq V \)." This deviates from the definition of Mc Cartan [5] only in the choice of the set \( V \); in the classic definition \( V \) is taken to be an open increasing neighborhood of \( x \).

In case of discrete order, Definition 3.11 coincides with the axiom \( T_{3-} \) for convergence spaces \( (F \in q(x) = \overline{F} \subseteq (tq)(x)) \), while Definition 3.12 coincides with \( T_{3} \) for convergence spaces \( (F \in q(x) = \overline{F} \subseteq q(x)) \). It follows that the topological modification preserves neither \( \text{ord } T_{3L} \) nor \( \text{st ord } T_{3L} \).

**THEOREM 3.13.** (cf. Mc Cartan [5, Remark 1]). In po convergence spaces, the axioms \( \text{ord } T_{1L} \) and \( \text{ord } T_{3L} \) together imply the axiom \( \text{ord } T_{2} \).

A convergence space \((X,q)\) is called strongly locally compact, if for every \( x \in X \) every \( F \in q(x) \) contains a coarser filter \( G \in q(x) \) which has a filter base of compact sets. In \( T_{2} \) topological spaces, it coincides with the usual definition of local compactness. We call a po convergence space \( \text{ord } T_{3} \), if it satisfies both \( \text{ord } T_{3L} \) and the dual axiom \( \text{ord } T_{3U} \). In a similar way we define \( \text{st ord } T_{3} \).

**THEOREM 3.14.** Let \((X,q,\leq)\) be a po convergence space, whose topological modification is \( \text{ord } T_{2} \). If \((X,q)\) is strongly locally compact, then \((X,q,\leq)\) is \( \text{st ord } T_{3} \).

**PROOF.** For \( x \in X \) and \( F \in q(x) \) there is a coarser filter \( G \in q(x) \) possessing a base of compact sets. The filter \( \bar{G} \) has a base of closed sets (cf. Nachbin [4, p. 44]). It follows

\[
\bar{1}(F) \subseteq \bar{1}(G) = i(G),
\]

which combined with the dual reasoning gives the theorem.

**COROLLARY 3.15.** (Mc Cartan [5, Th. 7]). Every po topological space, which is locally compact and \( \text{ord } T_{2} \), is also \( \text{ord } T_{3} \).

4. **SEPARATION AXIOMS AND CONNECTIVITY**

In this section, the axioms of Section 3 shall be related to connectivity, the concept of complete separatedness also being involved. The results to follow are new also in the theory of po topological spaces. For a corresponding study in topological spaces without order relation, reference is made to Preuss [6, Ch. 6]. Increasing continuous maps between po convergence spaces shall be called morphisms.

Let \( E \) denote a family of po convergence spaces. The elements \( x,y \) of an arbitrary po convergence space \( X \) are called \((X,E)\)-related, if \( \phi(y) \leq \phi(x) \) for all \( E \in E \) and all morphisms \( \phi : X \rightarrow E \), or if \( \phi(x) \leq \phi(y) \) for all \( E \in E \) and all morphisms \( \phi : X \rightarrow E \). Furthermore, the elements \( x,y \in X \) are called \((X,E)\)-identic, if \( \phi(x) = \phi(y) \) for all \( E \in E \) and all morphisms \( \phi : X \rightarrow E \).

**DEFINITION 4.1.** Let \( E \) be a family of po convergence spaces. A po convergence space \( X \) is called \( E \)-orderconnected, if for every \( x,y \in X \)

\[
x \parallel y = x \quad \text{and} \quad y \quad \text{are} \quad (X,E)\text{-related}
x \leq y = x \quad \text{and} \quad y \quad \text{are} \quad (X,E)\text{-identic}.
\]

In the special case discrete order, topological space (in both the po conver-
gence space $X$ and the family $E$), the definition above coincides with the
definition of $E$-connectedness for topological spaces in Preuss [6, Ch. 6].

**Remark 4.2.** A po convergence space $X$ is called strongly $E$-orderconnected, if
every $x, y \in X$ are $(X, E)$-identic. This is the natural definition of a connectedness
concept in po convergence spaces. Burgess and Mc Cartan [13] used a variant of this
definition in po topological spaces. Here Definition 4.1 shall be used, since from
our point of view it provides the best application. However, a short comparison of
the two definitions is called for. They coincide, if the po structure of $X$ is
directed (without restriction on the family $E$). In general, the two definitions do
not coincide. The stronger definition is applied in an example in Section 5.

**Definition 4.3.** Let $E$ be a family of po convergence spaces. A po convergence
space $X$ is called completely $E$-separated, if for every pair $x \nless y$ in $X$ there is
a space $E \in E$ and a morphism $\phi : X \rightarrow E$ for which $\phi(x) \nless \phi(y)$.

**Remark 4.4.** If $X$ is a po topological space and $E = \{[0,1]\}$, then Definition
4.3 coincides with Nachbin's definition of completely separated po topological space.

**Remark 4.5.** The condition of Definition 4.3 can be restated in the following
way: For every pair $x \parallel y$ in $X$ there are spaces $E, F \in E$ and morphisms
$\phi : X \rightarrow E$, $\psi : X \rightarrow F$ for which $\phi(x) \nless \phi(y)$ and $\psi(y) \nless \psi(x)$, and furthermore, for
every pair $x < y$ in $X$ there is a space $G \in E$ and a morphism $\eta : X \rightarrow G$ for
which $\eta(x) < \eta(y)$. Thus, the concept of completely $E$-separated is a natural
disconnectedness concept related to Definition 4.1.

**Remark 4.6.** In the special case discrete order and topological space,
Definition 4.3 coincides with the definition of totally $E$-connectedness topological
space (total $E$-zusammenhangsloser topologischer Raum) in Preuss [6, Ch. 6].

For $E$ a given family of po convergence spaces, the family of completely
$E$-separated po convergence spaces is denoted by $Q(E)$. It will play a crucial rôle
as a key, when translating the lower separation axioms of Section 3 into connectedness
concepts.

In the category of po convergence spaces and increasing continuous maps,
products and subspaces are formed in the obvious way. It is easy to prove

**Theorem 4.7.** For any family $E$ of po convergence spaces, the related family
$Q(E)$ is closed under formation of products and subspaces.

Denote the family of all $T\alpha_1$-ordered po convergence spaces by $\underline{\text{ord}} T\alpha_1$. In case
of $\alpha = 1L$, $1U$, 1 and 2, we shall determine at least one family $E_1$ of po convergence
spaces for which $\underline{\text{ord}} T\alpha_1 = Q(E_1)$. In case of $\alpha = 3L$, $3U$ and 3, we shall later
define another disconnectedness concept through which the regularity axioms shall be
represented.

**Theorem 4.8.** For $\alpha = 1L$, $1U$, 1 and 2, we have $\underline{\text{ord}} T\alpha_1 = Q(\underline{\text{ord}} T\alpha_1)$.

**Proof.** The theorem is proved for the case $\alpha = 1L$. Start by taking a po
convergence space $X \in Q(\underline{\text{ord}} T\alpha_1)$ and choose a $\neq b$ in $X$. There is a space
$E \in \underline{\text{ord}} T\alpha_1$ and a morphism $\phi : X \rightarrow E$ for which $\phi(a) \neq \phi(b)$. Hence, for any
filter $H \rightarrow \phi(a)$ there is a set $H \in H$ such that $h \neq \phi(b)$ for all $h \in H$.
Suppose there exists $F \rightarrow a$ such that every $F \in F$ contains some element $f$ with
$f \leq b$. Since $\phi(F) \rightarrow \phi(a)$, a contradiction is obtained, and hence $X \in \underline{\text{ord}} T\alpha_1$. 
Then take $X \in \text{ord } T_{1L}$. All $a \preceq b$ in $X$ are nicely separated by the identity map $X \rightarrow X \in \text{ord } T_{1L}$, and hence $X \in Q(\text{ord } T_{1L})$.

**COROLLARY 4.9.** The separation axioms $\text{ord } T_i$ $(i = 1L, 1U, 1, 2)$ are closed under formation of products and subspaces in the category of po convergence spaces.

In any representation $\text{ord } T_i = Q(E_i)$ it always holds $E_i \subseteq \text{ord } T_i$, but it is not necessary for $E_i$ to equal the whole family $\text{ord } T_i$ $(i = 1L, 1U, 1, 2)$. Endow the ordered set $\{1,2\}$ with the topology whose only non-trivial open set is $\{2\}$ ($\{1\}$), and denote the resulting po topological space by $S_{1L}$ ($S_{1U}$). Furthermore, let $E_1$ denote the family of all po topological spaces carrying interval topology.

**THEOREM 4.10.** The following representations hold: $\text{ord } T_{1L} = Q(S_{1L})$, $\text{ord } T_{1U} = Q(S_{1U})$ and $\text{ord } T_1 = Q(E_1)$. Moreover, neither $\text{ord } T_1$ nor $\text{ord } T_2$ can be interpreted using one-space families $E$.

**REMARK 4.11.** The ideas above are now applied on a new, weak separation axiom for po convergence spaces. We say a space $X$ is $T_0$-ordered, if for every pair $a \preceq b$ in $X$ at least one of the following conditions holds:

1. For every $F \ni a$ there is a set $F \in F$ such that $x \not\preceq b$ for all $x \in F$.
2. For every $G \ni b$ there is a set $G \in G$ such that $a \not\preceq y$ for all $y \in G$.

We denote this separation axiom by $\text{ord } T_0$. Obviously, $\text{ord } T_0 = Q(E_0)$, where $E_0 = (S_{1L}, S_{1U})$, and hence $\text{ord } T_0$ is closed under formation of products and subspaces in the category of po convergence spaces. A po convergence space is $T_0$-ordered, if and only if its topological modification is. In case of discrete order, $\text{ord } T_0$ coincides with the usual separation axiom $T_0$ for convergence spaces.

We proceed to the regularity axioms $\text{ord } T_i$ $(i = 3L, 3U, 3)$, starting with the definition of a new disconnectedness concept for po convergence spaces. Let $M$ be a subset and $p$ an element of the po convergence space $X$. By $M \ll p$ is indicated that there is a closed decreasing set $D$ in $X$ containing $M$ but not $p$.

Now, for $E$ a given family of po convergence spaces, let $R_L(E)$ be the family of po convergence spaces defined through

$$X \in R_L(E) \iff \text{For every closed decreasing set } D \subseteq X \text{ and for every } p \preceq D \text{ there is a space } E \in E \text{ and a morphism } \phi : X \rightarrow E \text{ for which } \phi(D) \ll \phi(p).$$

**THEOREM 4.12.** The representation $\text{ord } T_{3L} = R_L(\text{ord } T_{3L})$ holds. There is no po convergence space $E_{3L}$ for which $\text{ord } T_{3L} = R_L([E_{3L}])$.

**REMARK 4.13.** It is obvious how to define families $R_i(E)$, in order to have $\text{ord } T_i = R_i(\text{ord } T_i)$, $i = 3U, 3L$. (These regularity axioms were defined in the remarks preceding Theorem 3.14).

**REMARK 4.14.** Finally, we wish to point out that the results 4.7 - 4.10 and 4.12, although stated for po convergence spaces, also hold true for po topological spaces. For topological spaces without order, these results were presented in Preuss [6, Ch. 6].

5. **GENERATING NEW SEPARATION AXIOMS**

In Theorem 4.8 it was shown that the lower separation axioms of Section 3 are related to the connectivity concept given in Definition 4.1. In two examples, we
shall study deviating connectivity definitions, and then generate separation axioms
matching these definitions.

**EXAMPLE 5.1.** Definition 4.1 is strong in the sense that the corresponding
disconnectedness concept (Definition 4.3) allows a very weak \((X,E)\)-separation of
non-related elements (cf. Remark 4.5). Therefore, we mention the following pos-
sibility:

**Definition.** Let \(E\) be a family of po convergence spaces. A po convergence space
\(X\) is called weakly \(E\)-orderconnected, if for every \(x,y \in X\), every \(E \in E\) and every
morphism \(\phi : X \to E\) holds
\[
\begin{align*}
x \mathrel{||} y &= \phi(x), \phi(y) \text{ are order related} \\
x \leq y &= \phi(x) = \phi(y).
\end{align*}
\]
We introduce the corresponding disconnectedness concept \(Q'(E)\) by
\[
X \in Q'(E) \iff \text{For every } x \mathrel{||} y \text{ in } X \text{ there is a space } E \in E
\text{ and a morphism } \phi : X \to E \text{ such that } \phi(x) \mathrel{||} \phi(y), \text{ and furthermore,}
\text{for every } x < y \text{ in } X \text{ there is a space } F \in E \text{ and a morphism}
\phi : X \to F \text{ such that } \phi(x) < \phi(y).
\]
In the special case discrete order and topological space, these definitions
coincide with the definitions of \(E\)-connected and totally \(E\)-connectedness topological
spaces, respectively (cf. Preuss [6, Ch. 6]).

It can be proved that \(\text{ord } T_i = Q'(\text{ord } T_i)\). If \(E_i\) is a proper subfamily of
\(\text{ord } T_i\), then in general \(Q'(E_i)\) is a proper subfamily of \(Q(E_i)\). Thus, the family
\(Q'(E_i)\) defines a stronger separation axiom than \(Q(E_i)\), \(i = 1L, 1U, 1, 2\). If \(Q\)
is replaced by \(Q'\), Theorem 4.10 holds with the only exception that the space
\(S_{1L}(S_{1U})\) must be replaced by the po topological space \(E_{1L}(E_{1U})\). Hereby, \(E_{1L}(E_{1U})\)
is defined on the set \(\{a,b,c\}\), where the order is \(a\mathrel{||} b\), \(a \leq c\), \(b \leq c\)
\((a\mathrel{||} b, c \leq a, c \leq b)\) and the non-trivial open sets are \(\{c\}\), \(\{a,c\}\), \(\{b,c\}\) in both
cases.

**EXAMPLE 5.2.** Starting with the connectivity definition of Remark 4.2 (i.e.
strong \(E\)-orderconnectedness), we write for any po convergence space \(X\)
\[
X \in Q''(E) \iff \text{For every } x \neq y \text{ in } X \text{ there is a space } E \in E
\text{ and a morphism } \phi : X \to E \text{ for which } \phi(x) \neq \phi(y).
\]
For \(E\) a family of po convergence spaces, in general, \(Q(E)\) is a strict subfamily of
\(Q''(E)\), and hence \(Q''(E)\) defines a weaker separation axiom than \(Q(E)\). For instance,
the family \(Q((S_{1L}))\), i.e. \(\text{ord } T_{1L}\), is a strict subfamily of \(Q''((S_{1L}))\). We consider
\(Q''((S_{1L}))\) to be a separation axiom. A po convergence space is \(Q''((S_{1L}))\), if and
only if the topological modification is. A po topological space \(X\) is \(Q''((S_{1L}))\),
if and only if for every \(x \mathrel{||} y \in X\) there is an increasing open neighborhood of at
least one of the two elements which does not contain the other element, and
furthermore, for every \(x < y \in X\) there is an increasing open neighborhood of \(y\)
which does not contain \(x\).

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