SOME RADIUS OF CONVEXITY PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

KHALIDA I. NOOR AND HAILAH AL-MADIFER

Mathematics Department
Science College of Education for Girls
Sitten Road, Riyadh, Saudi Arabia

(Received April 21, 1983)

ABSTRACT. In this paper we consider some radius of convexity problems for certain classes of analytic functions. These classes, in general, are related with functions of bounded boundary rotation.

KEYWORDS AND PHRASES. Analytic functions, bounded boundary rotation, convex and close-to-convex functions. Univalent functions.

AMS(MOS) Subject Classification. Primary 30A32, Secondary 30A34.

1. INTRODUCTION.

Let \( V_k \) be the class of functions of bounded boundary rotation. Paatero [1] showed that a function \( f \), analytic in \( E = \{z:|z|<1\}, f(0) = 0, f'(0) = 1, \) \( f'(z) \neq 0; \) is in \( V_k \) if and only if, for \( z = re^{i\theta}, \)

\[
\int_0^{2\pi} \left| \frac{\Re \frac{zf'(z)}{f'(z)}}{\theta} \right| \, d\theta < k \pi
\]

It is geometrically obvious that \( k \geq 2 \) and \( V_2 \subset C \), the class of univalent convex function.

A class \( T_k \) of analytic functions related with the class \( V_k \) has been introduced and discussed in [2]. Let \( f \) with \( f(0) = 0, f'(0) = 1 \) be analytic in \( E \). Then \( f \in T_k, k \geq 2, \) if there exists a function \( g \in V_k \) such that, for \( z \in E, \)

\[
\Re \frac{f'(z)}{g'(z)} > 0
\]

It is clear that \( T_2 \subset K \), the class of close-to-convex functions introduced by Kaplan [3].

Let \( P_{a,n} \) denote the class of functions \( p(z) \) in \( E \) given by

\[
p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots, \quad n \neq 1, \text{ which satisfy the inequality}
\]
\[ |p(z) - \frac{1}{2a}| < \frac{1}{2a} , \quad 0 < a < 1 . \]

The class \( P_{\alpha,n} \) has been introduced in [4]. If \( \alpha = 0 \), the class \( P_{\alpha,n} \) reduces to the classical class of functions with positive real part.

We shall need the following results in the next section.

**Lemma 1.1 [4]**. Let \( p \in P_{\alpha,n} \), then for \( z \in E, |z| = r < 1 \)

(i) \[ \frac{1 - r^n}{1 + cr^n} < \frac{r^n}{p(z)} \leq \frac{1 + r^n}{1 - cr^n} \]

(ii) \[ \left| \frac{p'(z)}{p(z)} \right| < \frac{(1+cnr^n-1)}{(1+cr^n)(1-r^n)} , \]

where \( c = 1 - 2a \).

**Lemma 1.2 [5]**. If \( H \) and \( D \) are analytic in \( E \) and \( H(0) = D(0) = 0 \), and if \( D \) maps \( E \) onto many-sheeted region, which is starlike with respect to the origin, then \( \Re \frac{H}{D} > 0 \Rightarrow \Re \frac{H}{D} > 0, \quad z \in E \).

**Lemma 1.3 [6]**. Let \( g \in V_k \). Then \( G(z) = \frac{2}{z} \int_0^z g(t) dt \) is convex in the disc \( |z| < \frac{1}{2k} (k - \sqrt{2k+4}) \).

2. MAIN RESULTS

In all of the theorems, \( f \) and \( g \) will be analytic in \( E, f'(0) = 1, f(0) = 0 \). The univalence will not be assumed unless explicitly stated.

**Theorem 2.1.** Let \( g \in V_k \) and let \( \frac{f'(z)}{g'(z)} \in P_{\alpha,1} \). Then \( f \) maps \( |z| < r \) onto a convex domain, where \( r \) is the least positive root of

\[ cx^2 - x^2(cx+c) - x(k+1) + 1 = 0 . \quad (2.1) \]

**Proof:** Let \( \frac{f'(z)}{g'(z)} = p(z) \), \( p(z) \in P_{\alpha,1} \).

Then

\[ \frac{(zf'(z))'}{f'(z)} = \frac{zz'(z)'}{g'(z)} \cdot \frac{zp'(z)}{p(z)} . \]

Hence

\[ \Re \left( \frac{zf'(z)'}{f'(z)} \right) > \Re \left( \frac{zz'(z)'}{g'(z)} \right) - \left| \frac{zp'(z)}{p(z)} \right| . \quad (2.2) \]

Now, it is known [7] that if \( g \in V_k \), then

\[ \Re \left( \frac{zp'(z)}{g'(z)} \right) > \frac{r^2 - kr + 1}{1 - r^2} . \quad (2.3) \]

Using (2.3) and Lemma 1.1(ii) for \( n = 1 \), (2.2) becomes
Thus \( f \) is convex if the right hand side of (2.1) is positive.

**Corollary 2.1.** Let \( \alpha = 0 \) (\( c = 1 \)) which means \( \Re \frac{f''(z)}{g''(z)} > 0 \). Then \( f \) maps \( |z| < r = \frac{1}{k+2} \) onto a convex domain. This result was obtained in [2].

**Corollary 2.2.** For \( \alpha = \frac{1}{2} \), we have \( \frac{f''(z)}{g''(z)} - 1 \) for \( k = 3 \). Then \( f \) is convex for \( |z| < r = \frac{1}{k+1} \). For \( k = 4 \), \( V_k \) consists of univalent functions and \( r = \frac{1}{5} \).

This result is known [4].

**Corollary 5.3.** If \( \alpha = 0 \), and \( k = 2 \), then \( f \) maps \( |z| < r = 2 - \sqrt{2} \) onto a convex domain. This result is well-known [8].

**Remarks 2.1.** Let \( \alpha = 0 \) and \( k = 1 \). Then we obtain the known result \( r = 3 - 2\sqrt{2} \) of Ratti [9].

**Theorem 2.2.** Let \( g(z) \) and let \( \frac{f''(z)}{g''(z)} \in \mathbb{P}_{\alpha, 1} \). Then \( f \) maps \( |z| < r \) onto a convex domain where \( r \) is the least positive root of the equation

\[
\alpha x^3 - (k+2)x^2 - (k+1)x + 1 = 0.
\]

**Proof:** Let \( \frac{f''(z)}{g''(z)} = p(z) \), where \( p(z) \in \mathbb{P}_{\alpha, 1}, g(z) \in T_k \).

Then

\[
\Re \left( \frac{z f''(z)}{f'(z)} \right) > \Re \left( \frac{z g''(z)}{g'(z)} \right) - \left| \frac{z p''(z)}{p(z)} \right|
\]

For \( g(z) \), it is known [2] that

\[
\Re \left( \frac{z f''(z)}{f'(z)} \right) \geq \frac{r^2 - (k+2)r + 1}{1 - r^2}.
\]

Using (2.4) and Lemma 1.1(ii), we obtain the result.

**Corollary 2.4.** Let \( \alpha = \frac{1}{2} \) (\( c = 0 \)) and in this case \( f \) maps \( |z| < r = \frac{1}{k+3} \) onto a convex domain. The special case for \( k = 2 \) is known [4].

**Corollary 2.5.** For \( \alpha = 0 \) and \( k = 2 \), \( T_2 \subseteq K \) consists of close-to-convex univalent functions. Then the radius of convexity is \( r = 3 - 2\sqrt{2} \). This result is known [9].

**Theorem 2.3.** Let \( \Re \frac{f''(z)}{g'(z)} > 0 \) and \( \Re \frac{g''(z)}{g'(z)} > 0 \), where \( S \) belongs to the
class $S^*$ of starlike functions. Then $f$ maps $|z| < r = 4 - \sqrt{15}$ onto a convex domain.

**Proof:** We have

$$f'(z) = S'(z)h_1(z)h_2(z),$$

where $\text{Re } h_1(z) > 0$, $\text{Re } h_2(z) > 0$, $z \in \mathbb{C}$.

That is

$$\frac{\left(\frac{z}{f'(z)}\right)'}{f'(z)} = \frac{z}{S'(z)} + \frac{zh_1'(z)}{h_1(z)} + \frac{zh_2'(z)}{h_2(z)},$$

hence

$$\text{Re } \frac{\left(\frac{z}{f'(z)}\right)'}{f'(z)} > \frac{\text{Re } \left(\frac{z}{S'(z)}\right)'}{S'(z)} - \frac{zh_1'(z)}{h_1(z)} - \frac{zh_2'(z)}{h_2(z)}.$$  \hspace{1cm} (2.5)

Now it is well known [8] that for $S \in S^*$, $S \not\subset \mathbb{C}$,

$$\text{Re } \frac{\left(\frac{z}{f'(z)}\right)'}{f'(z)} > \frac{1 - 4r + r^2}{1 - r^2}.$$ \hspace{1cm} (2.6)

Also, if $\text{Re } h(z) > 0$, then it is known [16] that

$$\left|\frac{zh_1'(z)}{h(z)} + \frac{zh_2'(z)}{h(z)}\right| > \frac{3r}{1 - r^2}.$$ \hspace{1cm} (2.7)

Using (2.6) and (2.7), (2.5) yield:

$$\text{Re } \frac{\left(\frac{z}{f'(z)}\right)'}{f'(z)} > \frac{1 - 8r + r^2}{1 - r^2}.$$ \hspace{1cm} (2.8)

Hence $f$ is convex for $|z| < r = 4 - \sqrt{15}$.

**Theorem 2.4.** Let $\text{Re } \frac{f'(z)}{g'(z)} > 0$ and $\text{Re } \frac{S'(z)}{S'(z)} > 0$ where $S \in T_k$.

Then $f$ maps $|z| < r = \frac{(k+6) - \sqrt{(k+6)^2 - 4k}}{2}$ onto a convex domain.

The proof follows on the same lines of Theorem 2.3, by using (2.4).

**Corollary 2.6.** If $k = 2$, then $S \subset T_2 \subset T$. In this case $f$ maps $|z| < r = 4 - \sqrt{15}$ onto a convex domain.

**Theorem 2.5.** Let $f \in V_k$ and $f'(z) = \int_0^z (f'(t))^q dt$, $0 < q$. Then $f$ maps $|z| < r$ onto a convex domain, where $r$ is the least positive root of

$$(2a-1) x^2 - \alpha x + 1 = 0.$$ \hspace{1cm} (2.9)

**Proof:** We have $f'(z) = (f'(z))^q$, $0 < q$.

Thus

$$\frac{(zf'(z))'}{f'(z)} = a \left( \frac{zf'(z)}{f'(z)} \right) + (1-a).$$
Since $f \in \mathcal{V}_k$, using (2.3), we have
\[
\text{Re} \left( \frac{(zf')'(z)}{f'(z)} \right) > \frac{1-kr+r^2}{1-r^2} + (1-a) = \frac{(1-akr+2a-1)r^2}{1-r^2}
\]
and this gives us the required result.

**Theorem 2.6.** Let $f \in \mathcal{T}_k$ and $f(z) = \int_0^z (f'(t))^a \, dt$. Then $f_a$ maps $|z| < r$ onto a close-to-convex domain, where $r$ is the least positive root of (2.8).

**Proof:** Since $f \in \mathcal{T}_k$, there exists a $g \in \mathcal{V}_k$ such that $\text{Re} \left( \frac{f'(z)}{g'(z)} \right) > 0$. Let
\[
g_a(z) = \int_0^z [g'(t)]^{1/a} \, dt.
\]
Then
\[
(f_a'(z)/g_a'(z)) = (f'(z)/g'(z))^a.
\]
Using Theorem 2.5, it follows that $f_a$ is close-to-convex for $|z| < r$, where $r$ is the least positive root of (2.8).

**Corollary 2.7.** Let $f \in \mathcal{T}_k$, then $f_a$ is close-to-convex for $|z| < r$, where $r$ is the least positive root of
\[
(2a-1)x^2 - 4ax + 1 = 0.
\]
In this case, if $a = \frac{1}{2}$, then $f_a$ is close-to-close for $|z| < \frac{1}{2}$.

**Corollary 2.8.** Let $f \in \mathcal{T}_k$ and $a = \frac{1}{2}$. Then $f_a$ is close-to-convex for $|z| < r = \frac{2}{k}$. For $k=2$, we have a result proved in [11].

**Corollary 2.9.** For $k=2, f_a \in \mathcal{K}$, see [11].

**Theorem 2.7.** Let $f \in \mathcal{T}_k$ and $F(z) = \frac{2}{z} \int_0^z f(t) \, dt$. Then $F$ maps $|z| < r = \frac{1}{2} \left( k - \sqrt{k^2 - 4} \right)$ onto a close-to-convex domain.

**Proof:** Since $f \in \mathcal{T}_k$, there exists a $g \in \mathcal{V}_k$ such that $\text{Re} \left( \frac{f'(z)}{g'(z)} \right) > 0$. Let
\[
G(z) = \frac{2}{z} \int_0^z g(t) \, dt.
\]
We know, from Lemma 1.3, that $G$ is convex for $|z| < r = \frac{k - \sqrt{k^2 - 4}}{2}$. Now
\[
\frac{F'(z)}{G'(z)} = \frac{\left( \frac{2}{z} \int_0^z f(t) \, dt \right)'}{\left( \frac{2}{z} \int_0^z g(t) \, dt \right)'} = \frac{N}{D}
\]
and
\[
\frac{H'}{D'} = \frac{f'(z)}{g'(z)}.
\]
So \( \text{Re} \frac{N'}{D'} > 0 \). Applying Lemma 1.2 for \( |z| < r = \frac{k - \sqrt{k^2 - 4}}{2} \)
we have \( \text{Re} \frac{N}{D} > 0 \), which implies that \( F \) is close-to-convex for
\( |z| < r = \frac{k - \sqrt{k^2 - 4}}{2} \).

Corollary 2.10. When \( k=2 \), \( f(z) \subset T_2 \equiv K \) and hence \( F \subset K \) for \( z \in E \). This
result was obtained in [5] by Libera.

REFERENCES

1. PAATERO, V. Uber die Konforme Abbildung von Gebieten deren Rander
   von beschränkter Drehung Sind, Ann Acad. Sci. Fenn., Ser. A 33
   (1933), 77 pp.


4. SHAFFER, D.B. Radii of Starlikeness and Convexity for Special Classes
   73-80.

5. LIBERA, R.J. Some Classes of Regular Univalent Functions, Proc. Amer.

6. KARUJAKARAN, V. and PADMINA, K. Functions of Bounded Radius Rotation,

7. TAKAI, O. On certain Combinations for the Coefficients of Schlicht


9. RATTI, J.S. The Radius of Convexity of Certain Analytic Functions,

10. MACGREGOR, T.H. The Radius of Univalence of Certain Analytic Functions,