ALMOST-PERIODICITY IN LINEAR TOPOLOGICAL SPACES AND APPLICATIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

GASTON MANDATA N'GUEREKATA

Université de Bangui
Faculté des Sciences
BP 1450 Bangui
République Centrafricaine

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ABSTRACT. Let $E$ be a complete locally convex space (l.c.s.) and $f : \mathbb{R} \rightarrow E$ a continuous function; then $f$ is said to be almost-periodic (a.p.) if, for every neighbourhood (of the origin in $E$) $U$, there exists $\epsilon = \epsilon(U)>0$ such that every interval $[a, a+t]$ of the real line contains at least one point $\tau$ such that $f(t+\tau) - f(t) \in U$ for every $t \in \mathbb{R}$. We prove in this paper many useful properties of a.p. functions in l.c.s. and give Bochner's criteria in Fréchet spaces.

KEY WORDS AND PHRASES. Almost-periodic functions, Bochner's criteria, weakly almost-periodic functions, abstract differential equations, perfect Fréchet spaces, infinitesimal generator of equi-continuous $C_0$-group.

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1. INTRODUCTION

The notion of almost-periodic functions has been introduced by Bohl and Esclangon at the beginning of the century and widely studied by Bochner [1], [2] and many other mathematicians. The reader can see [3], [4], [5], [6], [7],... for what is written on the subject.

A definition of almost-periodic functions on a group and with values in a linear topological space is contained in the important 1935 paper of Bochner and Von Neumann [2]; we consider here the one suggested in [6] which is very easy to handle (see definition 1 below). Most of the results of Part I of this paper are known in Banach spaces. We give their extensions to linear topological spaces.

In Section 5 of our paper, we study almost-periodicity of solutions of some abstract differential equations of the form: $x'(t) = Ax(t) + f(t), -\infty < t < \infty$, in Fréchet spaces.

We suppose the reader is acquainted with elementary properties of linear topological spaces (see for example [8]).

We consider a locally convex space $E = E(\tau)$ over the field $\phi(\tau = \mathbb{R}$ or $\mathbb{C}$); its topology $\tau$ is generated by a family of continuous semi-norms $Q = \{p_q, q,\ldots\}$.

We assume $E$ is a Hausdorff space. A basis of neighbourhoods (of the origin in $E$) contains sets of the form $U = U(\epsilon; p_i, 1 \leq i \leq n) = \{x \in E; p_i(x) < \epsilon, i = 1, \ldots, n\} p_i \in Q$. $E$ is called a Fréchet space if $\tau$ is induced by an invariant and complete metric. If
E is a Fréchet space, we may take \( Q = \{ p_i \}_{i=1}^{\infty} \). A subset \( D \subseteq E \) is dense in \( E \) if every \( x \in E \) is the limit of a generalized sequence of elements of \( D \). A linear operator \( A : D(A) \to E \) with domain \( D(A) \) dense in \( E \) is closed if its graph \( G(A) \) is a closed subset of the product space \( E \times E \).

**THEOREM.** (See [9]). Let \( E \) be a complete locally convex space. Then the linear operator \( A : D(A) \to E \) is closed iff for every generalized sequence \( (x_\mu) \) in \( D(A) \) such that \( \lim \mu x_\mu = x \) and \( \lim \mu A x_\mu = y \) we have \( x \in D(A) \) and \( Ax = y \).

**COROLLARY.** Every continuous linear operator defined on all \( E \) is closed.

In a locally convex space \( E \), a subset \( X \) is called totally bounded if, for every neighbourhood (of the origin) \( U \), there corresponds a finite set \( Y \) such that \( X \subseteq Y + U \).

2. **ALMOST PERIODIC FUNCTIONS WITH VALUES IN A LOCALLY CONVEX SPACE.**

**DEFINITION 1.** Let \( E \) be a complete locally convex space (l.c.s.). A continuous function \( f : \mathbb{R} \to E \) is called almost-periodic (a.p.) if for each neighbourhood (of the origin) \( U \), there exists a real number \( \varepsilon = \varepsilon(U) > 0 \) such that every interval \( [a, a+\varepsilon] \) contains at least a point \( \tau \) such that \( f(t+\tau) = f(t) \in U \) for every \( t \in \mathbb{R} \).

Obviously \( \tau = \varepsilon(U) \) and we call it a \( U \)-translation number of the function \( f \). The following two theorems are known (see [6]). We give here a proof of the second one.

**THEOREM 1.** (a) If \( f : \mathbb{R} \to E \) is a.p., then \( f \) is uniformly continuous on \( \mathbb{R} \).

(b) If \( (f_n)_{n=1}^{\infty} \) is a sequence of a.p. functions which converge uniformly on \( \mathbb{R} \) to a function \( f \), then \( f \) is also a.p..

**THEOREM 2.** If \( f \) is a.p., then \( \{ f(t); t \in \mathbb{R} \} \) is totally bounded in \( E \).

**PROOF.** Let \( U \) be a given neighbourhood, and \( V \) a symmetric neighbourhood such that \( V + V \subseteq U \); let \( \varepsilon = \varepsilon(V) \) as in definition 1. By continuity of \( f \), the set \( \{ f(t); t \in [0, \varepsilon]\} \) is compact in \( E \) (see [8] proposition 7, p. 53). But in a l.c.s., every compact set is totally bounded (see [8] theorem 5, p. 60); therefore there exists \( x_1, \ldots, x_v \) such that for every \( t \in [0, \varepsilon] \), we have \( f(t) \in \bigcup_{j=1}^{v} (x_j + U) \).

Take an arbitrary \( t \in \mathbb{R} \) and consider \( \tau \in [-\varepsilon, -t+\varepsilon] \) a \( V \)-translation number of \( f \). Then we have:

\[
(f(t+\tau) - f(t)) \in V. \tag{2.1}
\]

Choose \( x_k \) between \( x_1, \ldots, x_v \) such that

\[
f(t+\tau) \in x_k + V. \tag{2.2}
\]

Let us write \( f(t) = x_k = [f(t) - f(t+\tau)] + [f(t+\tau) - x_k]. \) Then by (2.1) and (2.2) we get \( f(t) = x_k \in U \) and therefore \( f(t) \in x_k + U \); as \( t \) is arbitrary we conclude:

\[
\{ f(t); t \in \mathbb{R} \} \subseteq \bigcup_{j=1}^{v} (x_j + U). \]

The theorem is proved.

**REMARK 1.** If \( E \) is a Fréchet space, then \( \{ f(t); t \in \mathbb{R} \} \) is relatively compact in \( E \) if \( f \) is a.p.. For in every complete metric space, relative compactness and totally boundedness are equivalent ([13], p.13). We then conclude every sequence \( (f(t_n))_{n=1}^{\infty} \) contains a convergent subsequence.

**THEOREM 3.** Let \( E \) be a complete l.c.s. If \( f : \mathbb{R} \to E \) is a.p. then the functions \( \lambda f(\lambda t) \) and \( \tilde{f} \) defined by \( \tilde{f}(t) \equiv f(-t) \) are also a.p..

**PROOF.** \( \lambda f \) is obviously a.p.. Let us consider \( \tilde{f} \); by almost-periodicity of \( f \), if \( U \) is a given neighbourhood, there exists \( \varepsilon = \varepsilon(U) \) such that every interval \( [a, a+\varepsilon] \)
contains $\tau$ such that $f(t+\tau) - f(t) \in U$ for every $t \in R$. Put $s = -t$; we get:

$$\bar{f}(s-\tau) - \bar{f}(s) = f(-(s+\tau)) - f(-s) = f(t+\tau) - f(t).$$

Therefore $\bar{f}(s-\tau) - \bar{f}(s) \in U$ for every $s \in R$, which shows $f$ is a.p. with $-\tau$ as a U-translation number.

3. BOCHNER'S CRITERIA AND OTHER PROPERTIES.

We first give theorem 4 we prove as theorem 6.6 in [6].

**THEOREM 4.** Let $E$ be a Fréchet space and $f : R \to E$ a.p.; then for every real sequence $(s')_{n=1}^{\infty}$, there exists a subsequence $(s_n')_{n=1}^{\infty}$ such that $(f(t+s_n))_{n=1}^{\infty}$ is uniformly convergent in $t \in R$.

**PROOF.** Consider the sequence of functions $(f_{s_n})_{n=1}^{\infty}$ corresponding to $(s_n)_{n=1}^{\infty}$ and let $S = (s_n)_{n=1}^{\infty}$ be a dense sequence in $R$. By remark 1, we can extract from $(f(n+s_n))_{n=1}^{\infty}$ a convergent subsequence, for $(f(t); t \in R)$ is relatively compact in $E$.

Let $(f_{s_1,n})_{n=1}^{\infty}$ be the subsequence of $(f_{s_n})_{n=1}^{\infty}$ which converges at $n_1$. We apply the same argument as above to the sequence $(f_{s_{1,n}})_{n=1}^{\infty}$ to choose a subsequence $(f_{s_{2,n}})_{n=1}^{\infty}$ which converges at $n_2$. We continue the process and consider the diagonal sequence $(f_{s_{n,n}})_{n=1}^{\infty}$ which converges for each $n$ in $S$. Call this last sequence by $(f_{r_n})_{n=1}^{\infty}$. Now we are going to show it is uniformly convergent in $R$, i.e. for every neighbourhood $U$, there exists $N = N_U$ such that $f(t+r_n) - f(t+r_m) \in U$ for every $t \in R$ if $n, m > N$.

Consider an arbitrary neighbourhood $U$ and a symmetric neighbourhood $V$ such that $U = U + U + U + V$. Let $\varepsilon = \varepsilon(V)$ as in definition 1. By uniform continuity of $f$ over $R$ (theorem 1), there exists $\delta = \delta_V > 0$ such that

$$f(t) - f(t') \in V$$

for every $t, t' \in R$ with $|t - t'| < \delta$.

We divide the interval $[0, \varepsilon]$ into $\nu$ subintervals of length smaller than $\delta$.

Then, in each interval, we choose a point of $S$ and get $S_0 = \{\xi_1, \ldots, \xi_{\nu}\}$. As $S_0$ is finite, $(f_{r_n})_{n=1}^{\infty}$ is uniformly convergent over $S_0$; therefore there exists $N = N_0$ such that

$$f(\xi_{i+n} + r_n) - f(\xi_i + r_m) \in V$$

for every $i = 1, \ldots, \nu$ and if $n, m > N$.

Let $t \in R$ be arbitrary and $\tau \in [-\varepsilon, -\varepsilon + \varepsilon]$ such that $f(t+\tau) - f(t) \in V$. Choose $\xi_1$ such that $|t+\tau - \xi_1| < \delta$; then $f(t+\tau + r_{n}) - f(\xi_1 + r_{n}) \in V$, for every $n$. Therefore, if $n, m > N$, we get:

$$f(t+r_n) - f(t+r_m) \in U,$$

which proves uniform convergence of $(f(t+r_n))_{n=1}^{\infty}$.

To see (3.1) we write:

$$f(t+r_n) - f(t+r_m) = [f(t+r_n) - f(t+r+\tau)]$$

$$+ [f(t+r+\tau) - f(\tau+\xi_1 + r_{n})] + [f(\tau+\xi_1 + r_{n}) - f(\xi_1 + r_{m})]$$

$$+ [f(\xi_1 + r_{m}) - f(t+r_m+\tau)]$$

$$+ [f(t+r_m+\tau) - f(t+r_m)],$$

and we apply (3.2) or (3.3) to each term in square brackets. The theorem is proved.

We now state and prove Bochner's criteria:

**THEOREM 5.** Let $E$ be a Fréchet space. Then $f : R \to E$ is a.p. iff for every real...
sequence \((s'_n)_{n=1}^{\infty}\) there exists a subsequence \((s'_n)_{n=1}^{\infty}\) such that \((f(t+s'_n))_{n=1}^{\infty}\) converges uniformly in \(t \in R\).

**PROOF.** The condition is obviously necessary by theorem 4; let us show it is sufficient; suppose \(f\) is not a.p.; then there exists a neighbourhood \(U\) such that for every \(\ell > 0\), there exists an interval of length \(\ell\) which contains no \(U\)-translation number of \(f\), or:

there exists an interval \([-a,-a+\ell]\) such that for every \(t \in [-a,-a+\ell]\) there exists \(t = t_1\) such that \(f(t+t_1) - f(t) \notin U\).

Let us consider \(t_1 \in R\) and an interval \((a_1-b_1)\) with \(b_1-a_1 > 2|\tau_1|\) which contains no \(U\)-translation number of \(f\). Now let \(t_2 = \frac{a_2-b_2}{2}\); then \(t_2-t_1 \in (a_1,b_1)\) and therefore \(t_2-t_1\) cannot be a \(U\)-translation number of \(f\). Let us consider another interval \((a_2,b_2)\) with \(b_2-a_2 > 2(|\tau_1|+|\tau_2|)\), which contains no \(U\)-translation number of \(f\). Let \(t_3 = \frac{a_3-b_3}{2}\); then \(t_3-t_1, t_3-t_2 \in (a_2,b_2)\) and therefore \(t_3-t_1\) and \(t_3-t_2\) cannot be \(U\)-translation number of \(f\). We proceed and get a sequence \((\tau_n)_{n=1}^{\infty}\) such that no \(\tau_n\) is a \(U\)-translation number of \(f\);

\[
\frac{a_2-b_2}{2} - f(t+t_1, t_1) - f(t) \notin U. \quad (3.4)
\]

Put \(\sigma = \sigma_{mn} = t-t_n\); then \((3.4)\) becomes:

\[
f(\sigma+t_2) - f(\sigma+t_1) \notin U. \quad (3.5)
\]

Suppose there exists a subsequence \((\tau'_n)_{n=1}^{\infty}\) of \((\tau_n)_{n=1}^{\infty}\) such that \((f(t+\tau'_n))_{n=1}^{\infty}\) converges uniformly in \(t \in R\); then for every neighbourhood \(V\), there exists \(N = N_V\) such that if \(m, n > N\) (we may take \(m > n\)), then we have:

\[
f(t+\tau'_m) - f(t+\tau'_n) \notin V. \quad (3.6)
\]

for every \(t \in R\).

But this contradicts \((3.5)\); it suffices to take \(V = V\) and \(\sigma_{mn} = t_{mn}\) in \((3)\).

Therefore \((f(t+\tau'_n))_{n=1}^{\infty}\) does not contain any subsequence which converges uniformly in \(t\). The theorem is proved.

**REMARK 2.** Here we do not use metrizability of \(E\) in the proof of the sufficiency of the condition.

**THEOREM 6.** Let \(E\) be a Fréchet space and consider the functions \(f, g, f_1, f_2: R \to E\); then we have:

a) \(f + g\) is a.p. in \(E\) if \(f\) and \(g\) are a.p. in \(E\)

b) \(F = (f_1, f_2)\) is a.p. in the product space \(E \times E\) if \(f_1\) and \(f_2\) are a.p. in \(E\).

**PROOF.** It is very easy to prove a) and b) by using Bochner's criteria; we omit it.

The reader can see [9].

**COROLLARY 1.** If \(f_1\) and \(f_2\) are a.p. in the Fréchet space \(E\), then for every neighbourhood \(U\), \(f_1\) and \(f_2\) have common \(U\)-translation numbers.

**PROOF.** Let \(U\) be a given neighbourhood in \(E\); by theorem 6, the function \(f(t) = (f_1(t), f_2(t))\) is a.p.; consider now \(t\) a \(U\)-translation number of \(f\); then \(f(t+t) - f(t) \notin U \times U\), for every \(t \in R\) and therefore \(f_1(t+t) - f_1(t) \notin U\), \(i = 1, 2\), for every \(t \in R\); \(t\) is then a \(U\)-translation number of \(f_1\) and \(f_2\).

**REMARK 3.** Theorem 6, b) and corollary 1 are true even for \(n\) functions, \(n > 2\).
4. WEAKLY A.P. FUNCTIONS; INTEGRATION OF A.P. FUNCTIONS.

Let \( E \) be a complete locally convex space.

DEFINITION. A function \( f: \mathbb{R} \to E \) is called weakly a.p. (we write w.a.p.) in \( E \) if the numerical function \((xf(t))\) is a.p. for every \( x^* \in E^* \) where \( E^* \) is the dual space of \( E \).

Obviously every a.p. function is w.a.p.; and if \( f \) is w.a.p. then it is weakly continuous and weakly bounded.

THEOREM 7. Let \( E \) be a complete l.c.s. and \( f \) a w.a.p. and continous function; assume \( \{F(t); t \in \mathbb{R}\} \) is weakly bounded, where \( F(t) = \int_0^t f(o)do \); then \( F(t) \) is w.a.p..

PROOF. We first note existence of the integral because of continuity of \( f \) over \( \mathbb{R} \). Take any \( x^* \in E^* \); then \((xf(t))\) is a.p.. By continuity of \( x^* \), we have \((xf(F(t))) = \int_0^t (xf(o))do \), which is bounded by our assumption. Now \((xf(F(t)))\) is a.p. (see [6], theorem 6.20). The theorem is proved.

THEOREM 8. Let \( E \) be a Fréchet space and \( f: \mathbb{R} \to E \) a given function; then \( f \) is a.p. if \( f \) is w.a.p. and \( \{f(t); t \in \mathbb{R}\} \) is relatively compact in \( E \).

PROOF. The condition is obviously necessary. Let us show it is sufficient by contradiction. Suppose there exists \( t_0 \) such that \( f \) is discontinuous at \( t_0 \). Then we can find a neighbourhood \( U \) and two sequences \((s_{n_1})_{n=1}^{\infty} \) and \((s_{n_2})_{n=1}^{\infty} \) such that

\[
\lim_{n \to \infty} s_{n_1} = 0 = \lim_{n \to \infty} s_{n_2} \quad \text{and} \quad f(t_0 + s_{n_1}) - f(t_0 + s_{n_2}) \notin U \quad (4.1)
\]

for every \( n \in \mathbb{N} \). By relative compactness of \( \{f(t); t \in \mathbb{R}\} \), we can extract \((s_{n_1})_{n=1}^{\infty} \) and \((s_{n_2})_{n=1}^{\infty} \) from the respective first two sequences such that \( \lim_{n \to \infty} f(t_0 + s_{n_1}) = a_1 \in E \) and \( \lim_{n \to \infty} f(t_0 + s_{n_2}) = a_2 \in E \). Consequently, using (4.1), we get \( a_1 - a_2 \in E \) and therefore by the Hahn-Banach theorem ([13], corollary 1, p. 108), there exists \( x^* \in E^* \) such that \( x^*(a_1 - a_2) \neq 0 \); hence

\[
x^*(a_1) \neq x^*(a_2) \quad (4.2)
\]

By continuity of \( x^* \), we have:

\[
x^*(a_1) = \lim_{n \to \infty} x^*(f(t_0 + s_{n_1})) = \lim_{n \to \infty} x^*(f(t_0 + s_{n_2})) = x^*(a_2)
\]

which contradicts (4.2); \( f \) is therefore continuous over \( \mathbb{R} \).

We are now going to show almost-periodicity of \( f \); but first of all, we state and prove:

LEMA 1. Let \( E \) be a Fréchet space and \( \phi: \mathbb{R} \to E \) be a.p.. Let \((s_n)_{n=1}^{\infty} \) be a real sequence such that \( \lim_{n \to \infty} \phi^*(s_n) \) exists for each \( k = 1, 2, \ldots \), where \((n_k)_{k=1}^{\infty} \) is dense in \( \mathbb{R} \). Then \( \phi(t + s_n)_{n=1}^{\infty} \) is uniformly convergent in \( t \in \mathbb{R} \).

PROOF. Suppose by contradiction \( \phi(t + s_n)_{n=1}^{\infty} \) is not uniformly convergent in \( t \), then there exists \( n \in \mathbb{N} \) and \( t_n \in \mathbb{R} \) such that:

\[
\phi(t_n + s_{n_k}) - \phi(t_n + s_{n_l}) \notin U \quad (4.3)
\]
By Bochner's criteria we can extract two subsequences \((s'_m) \subset (s_m)\) and \((s'_n) \subset (s_n)\) such that
\[
\lim_{N \to \infty} \Phi(t + s'_n) = g_1(t) \quad \text{uniformly in } t \in \mathbb{R},
\]
\[
\lim_{N \to \infty} (t + s'_m) = g_2(t) \quad \text{uniformly in } t \in \mathbb{R}.
\]

Let \(V\) be a symmetric neighbourhood with \(V + V + V \subset U\). Then there exists \(N_0 = N_0 V\) such that if \(N > N_0\),
\[
\Phi(t + s'_n) - g_1(t_N) \in V,
\]
\[
\Phi(t + s'_m) - g_2(t_N) \in V.
\]

We conclude \(g_1(t_N) - g_2(t_N) \in V\). If not, we should get
\[
\Phi(t + s'_n) - \Phi(t + s'_m) = \Phi(t + s'_n) - \Phi(t + s'_m) - g_1(t_N) + g_2(t_N) - g_2(t_N) + g_2(t_N) - \Phi(t + s'_m)
\]
and therefore \(\Phi(t + s'_n) - \Phi(t + s'_m) \in U\); this contradicts (1).

We have found \(V\) with the property that if \(N\) is large enough, there exists \(t_N \in \mathbb{R}\) such that
\[
g_1(t_N) - g_2(t_N) \notin V.
\]
But this is impossible; because if we take a subsequence \((\xi_k)_{k=1}^{\infty} \subset (\eta_k)_{k=1}^{\infty}\) and 
\(\xi_k \to t_N\), then we have
\[
\lim_{N \to \infty} \Phi(\xi_k + s'_n) = \lim_{N \to \infty} \Phi(\xi_k + s'_m)
\]
for every \(k\), and therefore \(g_1(\xi_k) = g_2(\xi_k)\) for every \(k\); by continuity of \(g_1\) and \(g_2\),
\(g_1(t_N) = g_2(t_N)\), thus \(g_1(t_N) - g_2(t_N)\) belongs to every neighbourhood of \(0\). The lemma is proved.

Let us now continue proving theorem 8. Consider arbitrary real sequences 
\((h_n)_{n=1}^{\infty}\) and \((\eta_n)_{n=1}^{\infty}\) the rational numbers.

By relative compactness of \(\{f(t); t \in \mathbb{R}\}\), we can extract a subsequence \((h_n)_{n=1}^{\infty}\) (we do not change notation) such that for each \(r\),
\[
\lim_{n \to \infty} f(\eta_n + h_n) = x_r \quad \text{exists in } E. \quad (4.4)
\]

Now \((f(\eta_n + h_n))_{n=1}^{\infty}\) is uniformly convergent in \(r\). Suppose it is not; then we find a neighbourhood \(U\) and three subsequences 
\((\xi_r)_{r=1}^{\infty} \subset (\eta_r)_{r=1}^{\infty}\), \((h'_r)_{r=1}^{\infty} \subset (h'_r)_{r=1}^{\infty}\),
\((h''_r)_{r=1}^{\infty} \subset (h''_r)_{r=1}^{\infty}\) and
\[
f(\xi_r + h'_r) - f(\xi_r + h''_r) \notin U. \quad (4.5)
\]

By relative compactness of \(\{f(t); t \in \mathbb{R}\}\) we may say
\[
\lim_{r \to \infty} f(\xi_r + h'_r) = b' \in E, \quad (4.6)
\]
\[
\lim_{r \to \infty} f(\xi_r + h''_r) = b'' \in E,
\]
and using (4.5), we get
\[
b' - b'' \notin U. \quad (4.7)
\]

By the Hahn-Banach theorem, there exists \(x* \in E^*\) such that
\[
x*(b') \neq x*(b''). \quad (4.8)
\]
But \( f(t) \) is w.a.p. hence \((x^* f)(t)\) is a.p. and consequently it is uniformly continuous over \( R \).

Consider the sequence of functions \((\varphi_n)_{n=1}^\infty\) defined by:
\[
\varphi_n(t) = (x^* f)(t + h_n), \quad n = 1, 2, \ldots
\]
The equality \( \varphi_n(t + \epsilon) - \varphi_n(t) = x^* f(t + \epsilon + h_n) - x^* f(t + h_n) \) shows almost-periodicity of each \( \varphi_n \). Also \((\varphi_n)_{n=1}^\infty\) is equi-uniformly continuous over \( R \) because \((x^* f)\) is uniformly continuous over \( R \), as it is easy to see. Using (4.4), we can say
\[
\lim_{n \to \infty} x^* f(t + h_n) = x^* (x(t))
\]
for every \( r \). Therefore, by lemma 1, \((x^* f(t + h_n))_{n=1}^\infty\) is uniformly convergent in \( t \).

Consider now the sequences \((\xi_r + h'_r)_{r=1}^\infty\) and \((\xi_r + h''_r)_{r=1}^\infty\). By Bochner's criteria, we extract two subsequences (we use the same notations) such that \((x^* f(t + \xi_r + h'_r))_{r=1}^\infty\) and \((x^* f(t + \xi_r + h''_r))_{r=1}^\infty\) are uniformly convergent in \( t \in R \).

Let us now prove
\[
\lim_{r \to \infty} x^* f(t + \xi_r + h'_r) = \lim_{r \to \infty} x^* f(t + \xi_r + h''_r).
\]

Consider the inequality:
\[
|x^* f(t + \xi_r + h'_r) - x^* f(t + \xi_r + h''_r)|
\leq |x^* f(t + \xi_r + h'_r) - x^* f(t + \xi_r + h'_r)|
+ |x^* f(t + \xi_r + h''_r) - x^* f(t + \xi_r + h''_r)|
\]
\(r = 1, 2, \ldots\)

Let \( \epsilon > 0 \) be given; as \((x^* f(t + h_n))_{n=1}^\infty\) is uniformly convergent in \( t \), we choose \( \eta_\epsilon \) such that for \( r, s > \eta_\epsilon \), we have
\[
|x^* f(t + h_s) - x^* f(t + h_r)| < \frac{\epsilon}{2}, \quad \text{for } t \in R; \text{ then}
\]
for \( r, s > \eta_\epsilon \), we get
\[
|x^* f(t + \xi_r + h'_r) - x^* f(t + \xi_s + h'_s)| < \frac{\epsilon}{2}.
\]
Consequently, for \( r > \eta_\epsilon \), we get:
\[
|x^* f(t + \xi_r + h'_r) - x^* f(t + \xi_r + h'_r)| < \frac{\epsilon}{2},
\]
\[
|x^* f(t + \xi_r + h''_r) - x^* f(t + \xi_r + h'_r)| < \frac{\epsilon}{2}
\]
and the inequality (4.10) gives:
\[
|x^* f(t + \xi_r + h'_r) - x^* f(t + \xi_r + h''_r)| < \epsilon
\]
for \( t \in R \). (4.9) is then proved.

Now take \( t = 0 \); then using (4.6) we get:
\[
x^* (b') = \lim_{t \to 0} x^* f(\xi_r + h'_r) = \lim_{t \to 0} x^* f(\xi_r + h''_r) = x^* (b''),
\]
which contradicts (4.8) and uniform convergence in \( r \) for \((f(\eta_n + h_r))_{n=1}^\infty\).

If \( i, j > N \), we have
\[
f(\eta_n + h_{i+1}) - f(\eta_n + h_{j+1}) \in U, \quad \text{for every } r.
\]
Therefore if \( t \in R \), we take a subsequence of \((\eta_n)_{n=1}^\infty\) which converges to \( t \) and using continuity of \( f \) and the relation (4.11), we obtain, for \( i, j > N \),
\[
f(t + h_{i+1}) - f(t + h_{j+1}) \in U.
\]
f is then a.p..

**Theorem 9.** Let \( E \) be a Fréchet space. If \( f: R \to E \) is a.p. and \( \{F(t); t \in R\} \)
where \( F(t) = \int_0^t f(\sigma) \, d\sigma \) is relatively compact in \( E \), then \( F \) is a.p..
PROOF. Immediate from theorems 7 and 8.

THEOREM 10. Let $E$ be a complete l.c.s.. If $f$ is a.p. and its derivative $f'$ uniformly continuous over the real line, then $f'$ is also a.p..

PROOF. Consider the sequence of a.p. functions $\{n(f(t+\frac{1}{n}) - f(t))\}_{n=1}^{\infty}$; it suffices to prove it converges uniformly over the real line to $f'(t)$.

Let $U = U(c; p, 1 \leq i \leq n)$; by uniform continuity of $f'$, we can choose $\delta = \delta_U > 0$ such that $f'(t_1) - f'(t_2) < \epsilon$ for every $t_1, t_2$ with $|t_1 - t_2| < \delta$.

We write

$$f'(t) - n(f(t+\frac{1}{n}) - f(t)) = n \int_0^1 [f'(t) - f'(t+\sigma)]d\sigma.$$ 

Therefore, if we take $N = \frac{1}{\overline{U} \delta}$, then for $n > N$, we have:

$$p[f'(t) - n(f(t+\frac{1}{n}) - f(t))] \leq n \int_0^1 p[f'(t) - f'(t+\sigma)]d\sigma < \epsilon$$

for every semi-norm $p$ and every $t \in R$. The theorem is proved.

THEOREM 11. Let $E$ be a Fréchet space; then the set of all a.p. $f: R \rightarrow E$ is a Banach space under the supremum norm.

PROOF. Obvious; use theorems 1, 2 and 6.

5. APPLICATIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

A. A.P. SOLUTIONS OF $\left(\frac{d}{dt} - A\right)x = 0$

Consider in a complete l.c.s. $E$ the differential equation

$$\frac{dx}{dt} = Ax(t), \quad -\infty < t < \infty, \quad (5.1)$$

where $A$ is a continuous linear operator such that $\{A^k; \; k = 1, 2, \ldots\}$ is equi-continuous. A solution of (5.1) is a continuously differentiable function which satisfies (5.1).

It is easy to construct (as in [13] p. 244-246) a solution of the form:

$$e^{tA}x(0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x(0).$$

We can say more:

PROPOSITION 1. The function $e^{tA}x_0: R \rightarrow E$ is the unique solution of the Cauchy problem:

$$\frac{dx}{dt} = Ax(t), \quad -\infty < t < \infty, \quad (5.2)$$

$$x(0) = x_0.$$ 

PROOF. Suppose there exists another solution $y(t)$ with $y(0) = x_0$; consider the function $v(t) = e^{(t-t)A}y(t)$, with fixed $t$ and show it is constant over the real line; therefore $v(t) = v(0)$ for every $t \in R$, which means $v(t) = v(0)$, or $y(t) = e^{tA}x_0$, proving uniqueness of the solution (see [9] for a complete proof).

Now, define a perfect Fréchet space $E$ as a Fréchet space with the following property: every function $f: R \rightarrow E$ with (i) $\{f(t); t \in R\}$ is bounded in $E$; (ii) $f'(t)$ is a.p. in $E$; is necessarily a.p. in $E$.

We state and prove the two following theorems inspired from a result of Pekov (see [15] theorem 1.1) but they are not direct generalisations. In fact they are new results.

THEOREM 1. Let $E$ be a perfect Fréchet space; assume (i) $A$ is a compact linear operator; (ii) $\{A^k; k = 1, 2, \ldots\}$ is equi-continuous; (iii) for every semi-norm $p$, there exists a semi-norm $q$ such that $p[e^{tA}x] \leq q(x)$ for every $x \in E$ and every $t \in R$. 


Then every solution \( x(t) \) of (5.1) is a.p. in \( E \).

PROOF. Because \( x(t) = e^{tA}x(0) \), then \( x(t) \) is bounded in \( E \) by (iii). \( E \) being a perfect Fréchet space, it suffices to prove \( x'(t) \) is a.p.\

\([Ax(t); \; t \in \mathbb{R}] \) is also relatively compact in \( E \) for \( A \) is a compact operator;

consequently \( \{x(t); \; t \in \mathbb{R}\} \) is also relatively compact. Let \( (s^m)_{m=1}^{\infty} \) be an arbitrary

real sequence; we then can extract a subsequence \( (s^m_{n})_{n=1}^{\infty} \) such that \( (x'(s^m_{n}))_{n=1}^{\infty} \) is

cauchy sequence in \( E \). But we have:

\[
x'(t+s^m_{n}) = Ax(t+s^m_{n}) = Ae^{tA}x(s^m_{n}) = e^{tA}s^m_{n}Ax(s^m_{n}) = e^{tA}s^m_{n}Ax(s^m_{n})
\]

for every \( n = 1, 2, \ldots \), and every \( t \in \mathbb{R} \). If \( p \) is a given semi-norm, there exists a

semi-norm \( q \) such that

\[
p[x'(t+s^m_{n}) - x'(t+s^m_{m})] = p[e^{tA}(x(s^m_{n}) - x(s^m_{m}))] \\
\leq q(x(s^m_{n}) - x(s^m_{m}))
\]

for every \( t \in \mathbb{R} \) and every \( n, m \in \mathbb{N} \). Therefore \( (x'(t+s^m_{n}))_{m=1}^{\infty} \) is uniformly Cauchy in \( t \);

we then conclude almost-periodicity of \( x'(t) \) by Bochner's criteria.

THEOREM 2. Let \( E \) be a Fréchet space; assume conditions (1) - (iii) in theorem 1

are satisfied and moreover the range \( R(A) \) of \( A \) is dense in \( E \). Then every solution

\( x(t) \) of (5.1) is a.p. in \( E \).

We remark the first part of the proof of theorem 1 tells us if \( x(t) \) is a

solution of (5.1) with \( x(0) \in D(A) = E \), then \( x'(t) \) is a.p. Before proving Theorem

2 let us state and prove:

**Lemma 1.** Every solution of (5.1) with initial data in \( R(A) \) is a.p.

**Proof.** Let \( a \in R(A) \) and consider the solution \( y(t) \) with \( y(0) = a \); there exists

\( x_0 \in D(A) = E \) such that \( Ax_0 = a \). We have \( y(t) = e^{tA}a = e^{tA}x_0 = ae^{tA}x_0 = Ax(t) = \)

\( x'(t) \) where \( x(t) = e^{tA}x_0 \); therefore \( x'(t) \) (and consequently \( y(t) \)) is a.p.. The lemma

is proved.

**Proof of Theorem 2.** Consider a solution \( x(t) \) of (5.1) with \( x(0) \in E \); as \( R(A) \) is

dense in \( E \), there exists a sequence \( (a^m)_{m=1}^{\infty} \) in \( R(A) \) such that \( a \rightarrow x(0) \).

Consider a sequence of solutions \( (y(t))_{m=1}^{\infty} \) with \( y(0) = a_n \), \( n = 1, 2, \ldots \). To prove almost-

periodicity of \( x(t) \) it suffices to prove \( y(t) + x(t) \) uniformly in \( t \in \mathbb{R} \) for every

\( y_n(t) \) is a.p. by lemma 1. We have \( x(t) = e^{tA}x(0), y_n(t) = e^{tA}a_n, n = 1, 2, \ldots \). Now

given a semi-norm \( q \) there exists, by assumption (iii), a semi-norm \( r \) such that

\[
p(y_n(t) - x(t)) \leq q(a_n - x(0)), \quad \text{for every } t \in \mathbb{R} \text{ and every } n \in \mathbb{N}.
\]

The conclusion is immediate.

**B. A.F. SOLUTIONS OF** \( A \)

\[
\frac{dx}{dt} = Ax(t) + f(t), \quad -\infty < t < \infty
\]

(5.3)

where \( A \) is a closed linear operator with domain \( D(A) \) dense in a Fréchet space \( E \);

the function \( f(t) \) is a.p. in \( E \). Let us recall some useful definitions (see 13).

A family of continuous linear operators \( T(t), t \in \mathbb{R} \), is an equi-continuous

\( C_0 \)-group:
(i) \( T(t_1+t_2)x = T(t_1)T(t_2)x, \) for every \( x \in E \) and every \( t_1, t_2 \in \mathbb{R}; \)
(ii) for every semi-norm \( p \), there exists a semi-norm \( q \) such that \( p[T(t)x] \leq q(x) \)
for every \( x \in E \) and every \( t \in \mathbb{R} \).
(iii) \( \lim_{t \to t_0} T(t)x, \)
for every \( x \in E \) and every \( t_0 \in \mathbb{R}. \)

Now consider an equi-continuous \( C^0 \)-group \( T(t). \) \( A \) is called the infinitesimal generator of \( T(t) \) if \( Ax = \lim_{n \to 0} \frac{T(t)x - x}{n} \), i.e., \( A \) is the linear operator with domain \( D(A) = \{ x \in E; \lim_{n \to 0} \frac{T(t)x - x}{n} \text{ exists in } E \} \) and for every \( x \in D(A) \), \( Ax = \lim_{n \to 0} \frac{T(t)x - x}{n} \).

It can be proved \( \frac{d}{dt} T(t)x = AT(t)x = T(t)Ax \)
for every \( x \in D(A) \) (see[13] for the case of a semi-group).

We are going to prove the following theorem 3 which is a generalization of theorem 3.2 [15] due to ZAIDMAN.

**THEOREM 3.** Let \( E \) be a Fréchet space. Suppose \( x(t) \) is a solution of equation (5.3) with relatively compact trajectory; \( A \) is the infinitesimal generator of equi-
continuous \( C^0 \)-group \( T(t) \) such that \( T(t)x : \mathbb{R} \to E \) is a.p. for every \( x \in E; f(t) \) is a.p., Then \( x(t) \) is also a.p..

Before we prove theorem 3, let us mention some useful lemmas (see [9] for proofs):

**LEMMA 2.** Let \( E \) be a complete l.c.s. If \( f(t) \) is continuous, then \( T(t-o)f(o) : \mathbb{R} \to E \) is also continuous for every \( t \in \mathbb{R}. \)

**LEMMA 3.** In a complete l.c.s. \( E \), every solution of (5.3) admits the integral representation:
\[
x(t) = T(t)x(0) + \int_0^t T(t-o)f(o)\, d.
\]

**LEMMA 4.** Let \( E \) be a Fréchet space. If \( \{ T(t)x; t \in \mathbb{R} \} \) is relatively compact in \( E \)
for every \( x \in E \) and \( \{ f(t); t \in \mathbb{R} \} \) is also relatively compact in \( E \), then \( \{ T(t)f(t); t \in \mathbb{R} \} \) is relatively compact in \( E \).

**PROOF.** Let \( \{ t''_n \}_{n=1}^\infty \) be an arbitrary real sequence; by our assumption on \( f(t) \), we can extract a subsequence \( \{ t'_n \}_{n=1}^\infty \subset \{ t''_n \}_{n=1}^\infty \) such that \( \lim f(t'_n) \) exists in \( E \); let \( x \) be this limit.
Take another subsequence \( \{ t''_n \}_{n=1}^\infty \subset \{ t'_n \}_{n=1}^\infty \) such that \( \{ T(t''_n)x \}_{n=1}^\infty \) is a Cauchy sequence in \( E \). Write:
\[
T(t_n)f(t_n) - T(t_m)f(t_m) = [T(t_n) - T(t_m)] [f(t_n) - x] + \left[\frac{T(t_n) - T(t_m)}{t_n - t_m}\right] [f(t_n) - f(t_m)] + T(t_m)[f(t_m) - f(t_m)]
\]
\[
\text{Let } p \text{ be any semi-norm; then we have}
\]
\[
p[T(t_n)f(t_n) - T(t_m)f(t_m)] = p[\left[ T(t_n) - T(t_m) \right] [f(t_n) - x] + \left[\frac{T(t_n) - T(t_m)}{t_n - t_m}\right] [f(t_n) - f(t_m)] + T(t_m)[f(t_m) - f(t_m)]]
\]

Using equi-continuity of \( T(t) \), we can take a semi-norm \( q \) such that
\[
p[T(t_n)f(t_n) - f(t_m)] \leq q[f(t_n) - f(t_m)]
\]
and
\[
p[\left[ T(t_n) - T(t_m) \right] [f(t_n) - x]] \leq 2q[f(t_n) - x].
\]

Now we choose \( n \) and \( m \) sufficiently large such that
\[
q[f(t_n) - f(t_m)] < \frac{\epsilon}{3}, \quad q[f(t_n) - x] < \frac{\epsilon}{3}, \quad p[\left[ T(t_n) - T(t_m) \right] x] < \frac{\epsilon}{3}.
\]
then we obtain:
\[ p[T(t_n)f(t_n) - T(t_m)f(t_m)] < \varepsilon \]
which shows \( (T(t_n)f(t_n))_{n=1}^\infty \) is a Cauchy sequence. The lemma is proved.

**Lemma 5.** Let \( E \) be a Fréchet space and consider the equi-continuous \( C_0 \)-group \( T(t) \) such that \( T(t)x : R \to E \) is a.p. for every \( x \in E \). Suppose also \( f(t) \) is a.p.. Then \( T(t)f(t) : R \to E \) is a.p..

**Proof.** Consider \( U = U(c; p_i, 1 \leq i \leq n) \) a given neighbourhood; because of equi-
continuity of \( T(t) \), there corresponds to each semi-norm \( p_i \), a semi-norm \( q_i \) such that:

(i) \( p_i(T(t)x) \leq q_i(x), x \in E, t \in R \).

Consider also the symmetric neighbourhood
\[ V = V(\frac{c}{4}; q_i, 1 \leq i \leq n); V + V + V + V \subseteq U. \]

As \( \{f(t); t \in R\} \) is totally bounded, there exists \( t_1, \ldots, t_v \) such that for every \( t \in R \) we have \( f(t) \in \bigcup_{k=1}^v (f(t_k) + V) \). Consider now the following a.p. functions: \( f(t), T(t)f(t_k), k = 1, \ldots, v \). Then they have the same \( V \)-translation numbers; therefore we can say there exists \( \tau = \tau(V) > 0 \) such that any interval \([a, a + \tau]\) contains \( \tau \) with

\[ f(t+\tau) - f(t) \in V, t \in R \quad (5.4) \]

\[ T(t+\tau)f(t_k) - T(t)f(t_k) \in V, k = 1, \ldots, t \in R. \]

Take \( t \in R \) arbitrary; then there exists \( k (1 \leq k \leq v) \) such that
\[ f(t) \in f(t_k) + V \quad (5.5) \]

Write:

\[ T(t+\tau)f(t+\tau) - T(t)f(t) = \{T(t+\tau)[f(t+\tau) - f(t)] + \}
\]

\[ \{\{T(t+\tau)[f(t) - f(t_k)] + \{T(t+\tau)f(t_k) - T(t)f(t_k)] + \{T(t)f(t_k) - f(t)]} \}
\]

For every semi-norm \( p_i \); there exists a semi-norm \( q_i \) such that:
\[ p_i[T(t+\tau)f(t+\tau) - T(t)f(t)] \leq q_i[f(t+\tau) - f(t)]
\]

\[ + q_i[f(t) - f(t_k)] + p_i[T(t+\tau)f(t_k) - T(t)f(t_k)]
\]

\[ + q_i[f(t_k) - f(t)] \leq \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = \varepsilon, \quad (using \ (5.3) \ - \ (5.5)) \]

Therefore \( T(t+\tau)f(t+\tau) - T(t)f(t) \in U \) for every \( t \in R \), which is almost-periodicity

**Proof of Theorem 3.** By lemma 3 we have: \( x(t) = T(t)x(0) + \int_0^t T(t-\sigma)f(\sigma)d\sigma. \)

But \( T(t)x(0) \) is a.p.. It remains to prove the function
\[ v(t) = \int_0^t T(t-\sigma)f(\sigma)d\sigma \]

is also a.p..

As \( \{x(t); t \in R\} \) and \( \{T(t)x(0); t \in R\} \) are relatively compact, then \( \{v(t); t \in R\} \)
also is relatively compact. Let us write \( v(t) = \int_0^t T(t-\sigma)f(\sigma)d\sigma \)

\[ = T(t) \int_0^t T(-\sigma)f(\sigma)d\sigma. \]

Then \( T(-t)v(t) = \int_0^t T(-\sigma)f(\sigma)d\sigma. \)

By theorem 3 of chapter 1, \( T(-t)x \) is a.p. for every \( x \in E \), therefore \( \{T(-t)x; t \in R\} \)
is relatively compact for every \( x \in E \). By lemma 4, \( \{T(-t)v(t); t \in R\} \) and consequently \( \int_0^t T(-\sigma)f(\sigma)d\sigma; t \in R \) is relatively compact. By lemma 5, \( T(-t)f(t) \)
is a.p., therefore \( \int_0^t T(-\sigma)f(\sigma)d\sigma \) is a.p.. We apply again lemma 5 to conclude almost-
periodicity of \( \int_0^t T(t-\sigma)f(\sigma)d\sigma. \) Theorem 3 is proved.
THEOREM 4. Let E be a Fréchet space. Solutions of the equation \( x'(t) = Ax(t) \), \(-\infty < t < \infty\), with relatively compact trajectory are precisely almost-periodic ones, if A is the infinitesimal generator of equi-continuous \( C_0 \)-group \( T(t) \).

PROOF. Let \( x(t) \) be a solution of the given equation. It suffices to prove that if \( x(t) \) has a relatively compact trajectory, then \( x(t) \) is a.p.. Take an arbitrary real sequence \( (s'_n)_{n=1}^{\infty} \) we can extract a subsequence \( (s_n)_{n=1}^{\infty} \subset (s'_n)_{n=1}^{\infty} \) such that \( (x(n))_{n=1}^{\infty} \) is a Cauchy sequence in E; but we have

\[
x(t+s_n) = T(t+s_n)x(0) = T(t)x(s_n), \quad n = 1, 2, \ldots
\]

Therefore

\[
x(t+s_n) - x(t+s_m) = T(t)[x(s_n) - x(s_m)], \quad n, m \in \mathbb{N}.
\]

Let \( p \) be a given semi-norm; by equi-continuity of \( T(t) \), there exists a semi-norm \( q \) such that:

\[
p[x(t+s_n) - x(t+s_m)] \leq q[x(s_n) - x(s_m)], \quad t \in \mathbb{R}.
\]

Which shows \( (x(t+s_n))_{n=1}^{\infty} \) is a Cauchy sequence, uniform in \( t \in \mathbb{R} \). We conclude using Bohr's criteria.

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