A CONSTRUCTION OF THE GENERAL RELATIVISTIC BOLTZMANN EQUATION

P. DOLAN

Department of Mathematics
Imperial College
London S.W. 7

A.C. ZENIOS

Euritania 32
Athens 613, Greece
(Received December 4, 1983)

ABSTRACT. Our work depends essentially on the notion of a one-particle seven-dimensional state-space. In constructing a general relativistic theory we assume that all measurable quantities arise from invariant differential forms. In this paper, we study only the case when instantaneous, binary, elastic collisions occur between the particles of the gas. With a simple model for colliding particles and their collisions, we derive the kinetic equation, which gives the change of the distribution function along flows in state-space.

KEY WORDS AND PHRASES. General Relativistic Kinetic Theory, phase space, Boltzmann distribution function.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 73P, 83C.

NOTATION: The Latin a, b, c, d take the values 0, 1, 2, 3 and the Greek indices \(\kappa, \lambda, \mu\) the values 1, 2, 3. The symbols \(\varepsilon_{abcd}\) or \(\varepsilon_{\kappa\lambda\mu}\) are the totally antisymmetric ones. The symbols \(p^\kappa_{ab}\) are the Christoffel symbols of the space-time metric.

1. THE STATE-SPACE OF A PARTICLE

The state of a particle is represented by the pair \((x,p)\), where \(x = (x^0, x^1, x^2, x^3)\) is the position and \(p = (p^0, p^1, p^2, p^3)\) is the momentum of the particle. The oriented space-time manifold is denoted by \(X\) and its metric tensor \(\eta_{ab} = \eta_{ab}(x)\) has signature \(+---\). The momentum-space at a space-time point \(x\), denoted by \(P(x)\), is the local Minkowskian tangent plane to \(X\) at \(x\) [1, 2, 3, 4, 5].

We define the real non-negative map \(m\) on \(P(x)\) by

\[
m(p) = \sqrt{-\eta_{00}p_0}\eta_{ij}p^ip^j + 2\eta_{00}p^0p^\kappa\eta_{\kappa\lambda}p^\lambda .
\]

(1.1)

A particle having proper mass \(m_j\) is called a j-particle. The physical momentum-space of a j-particle at \(x\) is denoted by \(P_j(x)\) and is defined by

\[
P_j(x) = \{p \in P(x) : m(p) = m_j, p_0 = \epsilon_{0\alpha\kappa\lambda}(\epsilon_{\alpha\kappa\lambda}p^\kappa + \epsilon_{\kappa\lambda}p^\lambda ) > 0\}.
\]

(1.2)

Now, from (1.1) and (1.2), we obtain the relations [4]

\[
p_0 = \sqrt{-\eta_{00}m_j^2}\eta_{\kappa\lambda}p^\kappa p^\lambda
\]

(1.3)

\[
p^0 = \frac{-\eta_{00}p^\kappa + \sqrt{-\eta_{00}m_j^2}(\epsilon_{0\kappa\lambda}p^\kappa - \epsilon_{\kappa\lambda}p^\lambda )p^\lambda}{\eta_{00}}.
\]

(1.4)
The element of the volume in \( X \) and in \( P(x) \) is given respectively by \( \eta \) and \( \rho \), where

\[
\eta = \frac{\sqrt{-g(x)}}{4!} e_{abcd} dx^a_A dx^b_A dx^c_A dx^d_A = \sqrt{-g(x)} \ dx_1^A dx_2^A dx_3^A \quad (1.5)
\]

\[
\rho = \frac{\sqrt{-g(x)}}{4!} e_{abcd} dp^a_A dp^b_A dp^c_A dp^d_A = \sqrt{-g(x)} \ dp_1^\lambda dp_2^\lambda dp_3^\lambda \quad (1.6)
\]

The element of the volume in \( P_j(x) \) must be an invariant differential three-form, independent of \( dm \) and non-vanishing when \( m = 0 \). Differentiating (1.1) we obtain \( dp^0 = (m dm - p^\mu dp^\mu)/p_0 \), and substituting in (1.6) we have \( \rho = m m_\rho \), where \( m_\rho \) is obviously the required element of the volume \( [3] \) and it is given by

\[
\rho_j = \frac{\sqrt{-g(x)}}{p_0} \, p_1^a \, p_2^b \, p_3^c = \frac{\sqrt{-g(x)}}{3! p_0} \, e_{\lambda \mu \nu} p_\lambda^a \, p_\mu^b \, p_\nu^c . \quad (1.7)
\]

The state-space of the j-particle, denoted by \( M_j \), is a fibre bundle over \( X \), i.e. \( M_j = P_j(X) = U_j P_j(x) \). So, its element of the volume is given by

\[
\mathcal{W}_j = \eta \rho_j \quad (1.8)
\]

Before studying a gas, we consider a single j-particle, which moves in a given gravitational field. Its world-line, \( x(s_j) \), must be a geodesic in \( X \) \[2,3,4,5\]. So it moves according to the laws

\[
\frac{dx^a}{ds_j} = p^a \quad , \quad \frac{dp^k}{ds_j} = - \Gamma^k_{ab} p^b . \quad (1.9)
\]

The affine parameter \( s_j \) is the proper-time of the j-particle. The equations (1.9) define the vector field \( \frac{d}{ds_j} \) in \( M_j \) and the differential operator \( \frac{d}{ds_j} \) completely

\[
\frac{d}{ds_j} = p^a \frac{\partial}{\partial x^a} - \Gamma^a_{\mu \nu} p^\mu \frac{\partial}{\partial p^\nu} \quad , \quad d = p^a ds_j \frac{\partial}{\partial x^a} - \Gamma^a_{\mu \nu} p^\mu ds_j \frac{\partial}{\partial p^\nu} . \quad (1.10)
\]

The integral curves \( (x^a(s_j), p^\mu(s_j)) \) in \( M_j \) form the flow of the states in \( M_j \) generated by the vector field \( \frac{d}{ds_j} \). Physically the flow of the states represents the set of all possible trajectories of the j-particle in \( M_j \) \[3,4\].

The element of the volume in an hypersurface in \( M_j \), \( S \), non-tangential to \( u/ds_j \) must be an invariant differential six-form independent of \( ds_j \). Using (1.8) and (1.9) and putting

\[
\sigma_0 = \frac{\sqrt{-g(x)}}{3!} e_{abcd} dx^b_A dx^c_A dx^d_A \quad , \quad \tau_\kappa = \frac{\sqrt{-g(x)}}{2!} e_{\lambda \mu \nu} dp_\lambda^a \, dp_\mu^b \quad (1.11)
\]

we obtain

\[
\mathcal{W}_j = \omega_j ds_j . \quad (1.12)
\]

where \( \omega_j = \frac{1}{4} \, p^a_{\mu \nu} \, \eta_\mu \nu - \frac{1}{3} \, \Gamma^\kappa_{ab} \, p^a \, p^b \, \tau_\kappa \eta \) is obviously the required element of the volume.

Inserting (1.5) and (1.7) into (1.12) and applying the differential operator \( \frac{d}{ds_j} \), we have \( \frac{d}{ds_j} \mathcal{W}_j = 0 \). But, from (1.12) we have \( \frac{d}{ds_j} \mathcal{W}_j = \omega_j ds_j \), and since \( ds_j \neq 0 \) we can immediately see that \( \frac{d}{ds_j} \mathcal{W}_j = 0 \).
Applying the differential operator $\partial_x^a$ on $\sigma_a$ and using the known formula
\[
\frac{1}{\sqrt{-g(x)}} \partial_x^a \cdot \frac{\partial}{\partial x} \sigma_a = f_{ba},
\]
we obtain the relation $\sigma_a = f_{ba} \eta_a$.

2. THE DESCRIPTION OF THE RELATIVISTIC GAS

A gas in this paper consists of a large number of particles interacting through gravitational forces. We can obtain a relativistic description of such a system by the use of the state-space of one particle. Our description depends essentially on only two phenomena: (a) the collision of two particles, and (b) the motion of one particle in the intervals between collisions in the smoothed-out mean gravitational field, generated by all particles together.

A simple j-gas is the gas consisting of only j-particles. Here we are to study a gas which is the mixture of $N+1$ simple gases and in which only instantaneous, binary, elastic collisions occur between the particles. A binary elastic jk-collision is a collision by which a j-particle collides with a k-particle and the result is also a j-particle and a k-particle. The only collisions we can observe in $M_j$ are all the jk-collisions for $k = 0,1,...,N$. Due to elastic jk-collisions, j-particles are removed from their states in $M_j$ to other states in $M_j$. We say that the j-particle is annihilated at the state from which it is removed by the collision and a new j-particle created at the state to which it is removed by the collision. The newly created particle moves in $M_j$ on a trajectory defined by (1.9). This new moving particle may also be annihilated. The parts of a trajectory between creation and annihilation are called excited parts.

Let $S$ be a hypersurface in $M_j$, independent of $ds_j$. We say that a trajectory is an occupied trajectory with respect to $S$ if $S$ is crossed by an excited part of this trajectory once and only once.

The distribution function $[4]$ of the j-gas component of this mixture is denoted by $f_j(x,p)$, where $f_j(x,p)$ is a scalar function on the state-space of the j-particle $M_j$. If we denote by $N_j(S)$ the number of the occupied trajectories with respect to $S$, this number is obviously equal to the number of the j-particles crossing $S$ once and only once. So, we can define $N_j(S)$ by

\[
N_j(S) = \int_S f_j(x,p) \, \omega_j.
\]  

(2.1)

Now, consider the volume $D$ in $M_j$. We denote the boundary of $D$ by $\partial D$ and obviously $\partial D$ is a hypersurface in $M_j$. So, by (2.1), we have

\[
N_j(\partial D) = \int_{\partial D} f_j(x,p) \, \omega_j.
\]  

(2.2)

In the case when $\partial D = \partial_1 \cup \partial_2 D$, with $\partial_1 D \cap \partial_2 D = \emptyset$, and with $\partial_2 D$ lying in the future of $\partial_1 D$ (we say that $\partial_2 D$ lies in the future of $\partial_1 D$ if the vector field $d/s_j$ is directed from $\partial_1 D$ to $\partial_2 D$) we have

\[
N_j(\partial D) = N_j(\partial_1 D) - N_j(\partial_2 D).
\]  

(2.3)

Let us define:

\[
a_{jk}(D) = \text{the number of j-particles annihilated in } D \text{ due to jk-collisions},
\]
\[
c_{jk}(D) = \text{the number of j-particles created in } D \text{ due to jk-collisions},
\]
\[
a_j(D) = \sum_{k=0}^N a_{jk}(D), \quad c_j(D) = \sum_{k=0}^N c_{jk}(D).
\]

We obviously have
\[ a_j(D) - c_j(D) = \frac{N}{k=0} (a_{jk}(D) - c_{jk}(D)) \quad . \tag{2.4} \]

We will try to shed some light on the foundations upon which the whole theory is constructed by establishing the following theorem [5].

**THEOREM 1.** \( N_j(a(D)) = N_j(a(D)) - N_j(a(D)) = a_j(D) - c_j(D) \quad . \tag{2.5} \)

**PROOF.** \( N_j(a(D)) = \) (the number of j-particles crossing \( a_1 \) \( D \)) = (the number of j-particles crossing \( a_1 \) \( D \) and not crossing \( a_2 \) \( D \)) + (the number of j-particles crossing both \( a_1 \) \( D \) and \( a_2 \) \( D \)).

\( N_j(a(D)) = \) (the number of j-particles crossing \( a_1 \) \( D \) without crossing \( a_2 \) \( D \)) + (the number of j-particles crossing both \( a_1 \) \( D \) and \( a_2 \) \( D \)).

So: \( N_j(a(D)) - N_j(a(D)) = \) (the number of j-particles crossing \( a_1 \) \( D \) and not crossing \( a_2 \) \( D \)) - (the number of j-particles crossing \( a_1 \) \( D \) without crossing \( a_1 \) \( D \)) = (the number of j-particles crossing \( a_1 \) \( D \) and annihilated in \( D \)) - (the number of j-particles created in \( D \) and crossing in \( a_2 \) \( D \)) = \{ (the number of j-particles crossing \( a_1 \) \( D \) and annihilated in \( D \)) + (the number of j-particles which are both created and annihilated in \( D \)) \} - \{ (the number of j-particles created in \( D \) and crossing \( a_2 \) \( D \)) + (the number of j-particles which are both created and annihilated in \( D \)) \} = \( a_j(D) - c_j(D) \).

3. THE CONSTRUCTION OF THE RELATIVISTIC KINETIC EQUATION

Now, we are in the position to construct the relativistic kinetic equation. To achieve this we apply the theorem of Gauss to the relation (2.2), and analyse the mechanism of instantaneous, binary elastic collisions in the relation (2.4). The kinetic equation is the equation which gives the change of \( f_j(x,p) \) along the flow of the states in \( M_j \). This change is just the action of the vector field \( \nabla_j \) on \( f_j(x,p) \).

Applying the theorem of Gauss in (2.2) we have

\[
N_j(a(D)) = \int_{a(D)} f_j(x,p)w_j = \int_{a(D)} d(f_j(x,p)w_j) = \int_{a(D)} (df_j(x,D))w_j =
\]

\[
= \int_{a(D)} \frac{df_j(x,p)}{ds} ds \cdot w_j = \int_{a(D)} \frac{d}{ds}(f_j(x,p))w_j \quad . \tag{3.1} \]

Our aim now is to express \( a_j(D) - c_j(D) \) as a volume-integral over \( D \). We achieve this by the construction of a simple model for colliding particles. In doing this, we need to define the concepts of the j-range and the j-volume of the collision.

Before proceeding, we want to note that a binary, elastic collision does not occur at a space-time point, but it is an interaction between two particles located at different space-time points. When we say that such a collision is elastic and instantaneous we mean that (a) the interaction affects only the momenta of the colliding particles and not their space-time positions, and (b) the proper-duration of the interaction is zero. From these remarks, we can see that the collision
occurs on an hypersurface $U$ of $X$, defined by $d{s}_j = 0$, i.e. both particles must lie on an hypersurface upon which $s_j = \text{constant}$, while interacting.

The element of the volume on such an $U$ is given by

$$n_j = \frac{1}{4} p^{a} \sigma_{a} . \tag{3.2}$$

This is obvious because we have $n = \frac{\sqrt{g(x)}}{4!} e_{abcd} dx^{a}_A dx^{b}_A dx^{c}_A dx^{d} = \frac{1}{4} dx^{a}_A \sigma_{a} = ds_j \left( \frac{1}{4} p^{a} \sigma_{a} \right)$.

The particles constituting the gas, will be thought of as interacting through an inter-particle potential. Let us study the $jk$-collision by which the $j$-particle at $x$ with momentum $p$ emerges with momentum $p'$, after colliding with the $k$-particle at $y$ with momentum $q$. To examine the phenomenon we restrict our study to the approximation $y = x$. This restriction will be seen not to violate the generality, in view of a very natural assumption (4) concerning momenta at the same space-time point to be introduced later. The law of momentum-conservation is then valid, and is given by: $p - q = p' - q'$. So, $q' = p - q - p'$, and we can describe the collision by using only $p$, $q$, $p'$. The range of interaction of the $j$-particle for this collision, called $j$-range of the collision, must (a) be a neighbourhood of $x$ in $U$, and (b) be independent of $x$ and (c) depend on $p$, $q$, $p'$ continuously. So, the $j$-range must be given by means of a continuous, positive function of $p$, $q$, $p'$. Let us denote this function by $\sigma_{jk}(p,q,p')$. Now, since we know the element of the volume of $U$, namely $n$ we can put:

$$j\text{-range of the collision} = \sigma_{jk}(p,q,p') n_j . \tag{3.3}$$

We say that the $jk$-collision occurs, when the $k$-particle lies inside the $j$-range of the collision and that the $jk$-collision occurs in propertime $ds_j$ when the $k$-particle finds itself inside the $j$-range of the collision in proper-time $ds_j$.

Since we know $p$ and $q$ we can find the relative velocity between the two particles. The relative velocity is given by the formula [4]

$$v_{pq} = \sqrt{\frac{\sigma_{jk}(p,q,p') n_j}{\sigma_{jk}(p,q,p') n_j}} . \tag{3.4}$$

where $(p,q) = g^{ab} p^{a} q^{b}$ .

The relative velocity multiplied by $ds_j$ gives the proper-distance between the two particles, and since $ds_j$ is perpendicular to the $j$-range of the collision we have:

$$\text{proper-distance of k-particle from j-range} = v_{pq} ds_j . \tag{3.5}$$

Multiplying (3.3) with (3.5), we obtain a volume which we call $j$-volume of the collision. In fact $v_{pq} ds_j \sigma_{jk}(p,q,p') n_j = v_{pq} \sigma_{jk}(p,q,p') n_j$. Now, using the concept of the $j$-volume of the collision we can say that the $jk$-collision occurs in proper-time $ds_j$ when the $k$-particle lies inside the $j$-volume of the collision.

For convenience we make the following assumption: For every $y$ lying inside the $j$-volume of the collision, we have $f_k(y,q) = f_k(x,q)$. (*)

The collision-number assumption or Boltzmann's "Stosszahlenaetz", in our case can be expressed by means of the following propositions: (a) The gas is sufficiently dilute so that each $j$-volume contains at most one $k$-particle with momentum $q$. 

The existence of such k-particles leads to jk-collisions. (b) The number of k-particles, with momentum qρk, lying inside the total volume of the j-volumes is assumed to be equal to the number obtained by multiplying the total volume of the j-volumes with the number of k-particles with qρk, lying in ρk. Now

\[ \text{number of j-particles with } p \rho_j, \text{ lying in the j-range of the collision} = f_j(x,p) \rho_j \Lambda(\sigma_jk(p,q,p')\eta_j) \] (3.6)

and

\[ \text{number of k-particles with } q \rho_k, \text{ of which the proper-distance from the j-range is smaller than } v_{pq} ds_j = f_k(x,q) \rho_k \Lambda(v_{pq} ds_j) . \] (3.7)

**Theorem 2.** The number of the jk-collisions occuring in proper-time ds_j, with p_ρ_j, q_ρ_k resulting in p_ρ'_j is given by multiplying (3.6) with (3.7) and with ρ'_j, i.e. this number is given by

\[ f_j(x,p)f_k(x,q)v_{pq}\sigma_{jk}(p,q,p')\rho_{j'k'}\Lambda_{j'k'} \] . (3.8)

(In arriving at (3.8) we have used (1.8) and (3.2)).

**Proof.** The number of j-particles with p_ρ_j is f_j(x,p)ρ_j. Imagine that to each of these particles there is attached one j-volume. The number of such j-volumes is obviously f_j(x,p)ρ_j. So the total volume occupied by the j-volumes is

\[ f_j(x,p)\rho_j \Lambda(v_{pq}\sigma_{jk}(p,q,p'))\eta_j = f_j(x,p)v_{pq}\sigma_{jk}(p,q,p')\eta_j . \] Using this result, combining the assumptions (a) and (b) of the "stosszahlansatz" and multiplying with ρ'_j we obtain the number which is the one given by (3.8).

Integrating (3.8) over D, we obtain the number of collisions annihilating j-particles inside D. But this number is just the number of annihilations inside D due to jk-collisions, i.e. a_{jk}(D). So,

\[ a_{jk}(D) = \int_D \left( \int_{P'_j(x)} \left( \int_{P_k(x)} f_j(x,p)f_k(x,q)v_{pq}\sigma_{jk}(p,q,p')\rho_j \right) \right) \eta_j . \] (3.9)

By the same method we find that the number of creations inside D due to jk-collisions, i.e. c_{jk}(D), is given by the relation

\[ c_{jk}(D) = \int_D \left( \int_{P'_j(x)} \left( \int_{P'_k(x)} f_j(x,p')f_k(x,q')v_{p'q'}\sigma_{jk}(p',q',p)\rho_j \right) \right) \eta_j . \] (3.10)

Now, from (2.4) we have

\[ a_j(D) - c_j(D) = \sum_{k=0}^N (a_{jk}(D) - c_{jk}(D)) = \int_D I_j(x,p) \eta_j , \] (3.11)

where I_j(x,p), the collision-integral is defined by the relation

\[ I_j(x,p) = \sum_{k=0}^N \left( \int_{P'_j(x)} f_j(x,p) \left( \int_{P_k(x)} f_k(x,q)v_{pq}\sigma_{jk}(p,q,p')\rho_k \right) \rho_j - \int_{P'_j(x)} f_j(x,p') \left( \int_{P'_k(x)} f_k(x,q')v_{p'q'}\sigma_{jk}(p',q',p')\rho_k \right) \rho_j \right) . \] (3.12)
Now, from (2.5), (3.1) and (3.11) we have that
\[ \int_D \frac{d}{d\tau_j} (\mathbf{r}_j(x,p)) \cdot \mathbf{v}_j = \int_D I_j(x,p) \mathbf{v}_j \]
for every \( \mathbf{v} \) in \( \Pi_j \). So
\[ \frac{d}{d\tau_j} (\mathbf{r}_j(x,p)) = I_j(x,p) \quad . \] (3.13)

This is the kinetic equation [4].

When \( a_j(D) = c_j(D) = 0 \) (case of the free-equilibrium) and when \( a_j(D) \neq 0 \), \( c_j(D) \neq 0 \) with \( a_j(D) - c_j(D) = 0 \) (case of the balanced-equilibrium) we obviously have \( I_j(x,p) = 0 \) and so
\[ \frac{d}{d\tau_j} (\mathbf{r}_j(x,p)) = 0 \quad . \] (3.14)

REFERENCES


