THE REGULARITY SERIES OF A CAUCHY SPACE

DARRELL C. KENT

Department of Pure and Applied Mathematics
Washington State University
Pullman, Washington 99164-2930 U.S.A.

(Received December 16, 1983)

ABSTRACT. This study extends the notion of regularity series from convergence spaces to Cauchy spaces, and includes an investigation of related topics such as that $T_2$ and $T_3$ modifications of a Cauchy space and their behavior relative to certain types of quotient maps. These concepts are applied to obtain a new characterization of Cauchy spaces which have $T_3$ completions.

KEY WORDS AND PHRASES. Cauchy space, R-Series, Regular Cauchy space, $C_3$ Cauchy space, cauchy restriction, Wylor modification.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE 54E10, 54D10, 54A20.

0. INTRODUCTION.

In two earlier papers ([5] and [7]), the author and Gary Richardson introduced the regularity series (or R-series) of a convergence space. Because the ideas and results developed in these papers have been fruitful in later investigations, it seems appropriate to extend at least some of them to the realm of Cauchy spaces. This is especially natural in view of the key role which regularity plays in the theory of Cauchy space completions.

We shall make one significant deviation from the notation of [5] and [7]; a convergence space will be denoted by "$(X,q)$" (where $X$ is the underlying set and $q$ the convergence structure) rather than by "$X". The usual notation for a Cauchy space is "$(X,\mathcal{C})$", where $\mathcal{C}$ is the Cauchy structure. This will enable us to make an easy transition from a convergence structure to the associated Cauchy structure (or vice versa) on the same underlying set.

In Section 1, we define the R-series and W-series of an arbitrary Cauchy space and show that their properties are analogous to those of the R-series for a convergence space discussed in [7]. Section 2 considers certain Cauchy quotient maps relative to which the R- and W-series are well behaved. The next section considers the $T_2$ and $T_3$ modifications of a Cauchy space. In Section 4 we consider admissible convergence structures (those which admit a Cauchy structure); the interaction between admissibility and regularity is studied, and the symmetric
series for an arbitrary convergence space is introduced. Section 5 introduces a
generalization of the Wyler completion [2] which is called the Wyler modification.
The concluding section extends a set function originally defined in [4] and makes
uses of the results of preceding sections to characterize Cauchy spaces having $T_3$
completions.

1. THE CAUCHY R-SERIES.

Let $X$ be a set, $F(X)$ the set of all filters on $X$. The fixed ultrafilter
generated by $x \in X$ will be denoted by $\hat{x}$. If $F, G \in F(X)$ and $F \cap G \neq \emptyset$ for all
$F \in F$, $G \in G$, we shall write "$F \uplus G$".

The term convergence space will mean a "Limitierung" in the sense of Fischer
[1]. For any convergence space $(X, q)$, \(^{cl_q^n}\) denotes the $n$th iteration of the
$q$-closure operator. We shall also introduce a weak $q$-closure operator defined as
follows: If $A \subseteq X$, then $wcl_q A = \{ x \in X : \exists y \in X \text{ for some } y \in A \}$. The $n$th
iteration of the weak closure operator is denoted "wcl\(^n_q\)". A convergence space $(X, q)$ is
regular (respectively, weakly regular) if $cl_q F \to x$ (respectively, wcl\(^q\) $F \to x$)
whenever $F \to x$. A convergence space $T_2$ (respectively, $T_1$) if each filter converges to
at most one point (respectively, each fixed ultrafilter converges to exactly one
point).

Starting with a set $X$, a subset of $\mathcal{C}$ of $F(X)$ is called a Cauchy structure on
$X$ if the following conditions are satisfied:

\begin{align*}
(c_1) \quad & \hat{x} \in \mathcal{C}, \text{ for all } x \in X \\
(c_2) \quad & F \in \mathcal{C} \text{ and } F \cap G \in \mathcal{C} \text{ implies } G \in \mathcal{C} \\
(c_3) \quad & F, G \in \mathcal{C} \text{ and } F \uplus G \in \mathcal{C} \text{ implies } F \cap G \in \mathcal{C}.
\end{align*}

If $(X, \mathcal{C})$ is a Cauchy space, the induced convergence structure $q\mathcal{C}$ on $x$ is
defined by $F \to x$ if $F \cap \hat{x} \in \mathcal{C}$. A Cauchy space is regular (respectively, weakly
regular) if $F \in \mathcal{C}$ implies $cl_q F \in \mathcal{C}$ (respectively $wcl_q F \in \mathcal{C}$). A Cauchy space is
$T_2$ if $\hat{x} \cap \hat{y} \in \mathcal{C}$ implies $x = y$. A $T_2$ Cauchy space is obviously weakly regular; a $T_2$
regular Cauchy space (or convergence space) is said to be $T_3$. A Cauchy space $(X, \mathcal{C})$
is complete if each $F \in \mathcal{C}$ is $q$-convergent: if every ultrafilter in $F(X)$ belongs
to $\mathcal{C}$, then $\mathcal{C}$ is said to be totally bounded.

A function $f : (X_1, \mathcal{C}_1) \to (X_2, \mathcal{C}_2)$ is Cauchy continuous if $F \in \mathcal{C}_1$
implies $f(F) \in \mathcal{C}_2$; a Cauchy continuous function will henceforth be called a map. The
Cauchy structures on a set $X$ are partially ordered by the relation $\mathcal{C}_1 \leq \mathcal{C}_2$ iff the
identity function $id_X : (X, \mathcal{C}_2) \to (X, \mathcal{C}_1)$ is a map.

It is well known that, for any Cauchy space $(X, \mathcal{C})$ there is a finest regular
Cauchy structure $\mathcal{C}$ on $X$ which is coarser than $\mathcal{C}$; $\mathcal{C}$ is called the regular modifi-
cation of $\mathcal{C}$. Similarly, there is a finest weakly regular Cauchy structure $w\mathcal{C}$
coarser than $\mathcal{C}$ which is called the weakly regular modification of $\mathcal{C}$. Starting
with a Cauchy space $(X, \mathcal{C})$ we shall construct two series of Cauchy structures which
terminate, respectively, in the regular and weakly regular modifications of $\mathcal{C}$.

Every collection $A$ of subsets of $X$ generates a unique Cauchy structure in a
manner which we shall now describe. First, we say that a finite set of filters
\{F_1, \ldots, F_n\} is linked if they can be arranged (by renumbering, if necessary) in such a way that \(F_1 \vee F_2, F_2 \vee F_3, \ldots, F_{n-1} \vee F_n\). Starting with an arbitrary collection \(A\) of subsets of \(X\), define \(A' = A \cup \{x : x \in X\}\), and let \(C_A = \{G \in \mathcal{F}(X) : \text{there are linked filters } F_1, \ldots, F_n \text{ in } A' \text{ such that } G \geq \bigwedge_{i=1}^{n} F_i\}\).

**Proposition 1.1.** In the terminology of the preceding paragraph, \(C_A\) is the finest Cauchy structure on \(X\) which contains all members of the collection \(A\).

We shall refer to \(C_A\) as the **Cauchy structure generated by** \(A\).

The R-series \(\{r_\beta C\}\) for \((X, C)\) is constructed as follows:

\[ r_0 C = C \]
\[ r_1 C \text{ is the Cauchy structure on } X \text{ generated by } \{c_{\beta}^n F : F \in \mathcal{C}, n \in \mathbb{N}\} \]
\[ \vdots \]
\[ r_\beta C \text{ is the Cauchy structure on } X \text{ generated by } \{c_{\beta}^n F : F \in \mathcal{C}, n \in \mathbb{N}\} \]

If \(\beta\) is a non-limit ordinal where \(q_{\beta - 1}\) is the convergence structure compatible with \(r_{\beta - 1}\).

\[ r_\beta C = \bigcup_{\alpha < \beta} r_\alpha C \] if \(\beta\) is a limit ordinal.

The W-series \(\{w_\gamma C\}\) is obtained by repeating the preceding construction using the weak closure operator in place of the closure operator.

Let \(\nu C\) (respectively, \(\lambda C\)) be the least ordinal number \(\gamma\) such that \(r_\gamma C \in r_{\gamma + 1} C\) (respectively, \(w_\gamma C = w_{\gamma + 1} C\)). The next result resembles Proposition 2.1, 7; all parts of this proposition are either obvious or straightforward, and we omit the proof.

**Proposition 1.2.** Let \((X, C)\) be an arbitrary Cauchy space; let \(\beta, \gamma\) be ordinal numbers with \(\beta < \gamma\).

1. \(r_\beta C \leq r_\gamma C \leq r_\beta C \leq C\)
2. \(w_\beta C \leq w_\gamma C \leq w_\beta C \leq C\)
3. \(r_\gamma C \leq w_\gamma C\)
4. \(r_\beta C = r_\beta C \iff \gamma \geq \nu_\beta C\)
5. \(w_\gamma C = w_\gamma C \iff \gamma \geq \lambda_\gamma C\)

**Proposition 1.3.** If \((X, C)\) is a complete Cauchy space, then \((X, r_\beta C)\) and \((X, w_\beta C)\) are complete for all ordinal numbers \(\beta\).

**Proof.** Let \(G \in r_1 C\); then there are \(F_1, \ldots, F_n\) in \(C\) and \(m \in \mathbb{N}\) such that \(c_{\beta}^m F_1, \ldots, c_{\beta}^m F_n\) are linked and \(G \geq \bigwedge_{i=1}^{n} c_{\beta}^m F_i\). Since \((X, C)\) is complete, there are \(x_i, i = 1, \ldots, n\), such that \(F_i \cap \hat{x}_i \in C\). Thus \(G \cap \hat{x}_i \in r_1 C\) for \(i = 1, \ldots, n\), and it follows that \(r_1 C\) is complete. This reasoning extends by transfinite induction to all ordinal numbers \(\beta\).

**Proposition 1.4.** Let \(f : (X, C) \to (Y, D)\) be a map. In the following diagrams, in which all vertical arrows are \(f\) and all horizontal arrows are the respective identity functions, each arrow is a map.
(X, C) \to (X, r_\beta C) \to \ldots \to (X, r_{\beta_2} C) \to \ldots \to (X, r C)
\downarrow
\downarrow
\downarrow
(Y, D) \to (Y, r_\beta D) \to \ldots \to (Y, r_{\beta_2} D) \to \ldots \to (Y, r D)
\downarrow
\downarrow
\downarrow
(X, C) \to (X, w_\beta C) \to \ldots \to (X, w_{\beta_2} C) \to \ldots \to (X, w C)
\downarrow
\downarrow
\downarrow
(Y, D) \to (Y, w_\beta D) \to \ldots \to (Y, w_{\beta_2} D) \to \ldots \to (Y, w D)

PROOF. From the original assumption, it follows that $f(\text{cl}^n_q F) \supset \text{cl}^n_q f(F)$
for all $F \in \mathcal{F}(X)$. It follows easily that $f : (X, x_1 C) \to (Y, r_1 D)$ is a map. This
reasoning can be extended by induction to the remaining vertical arrows in the
first diagram. Similar reasoning can be applied to the second diagram. 

2. QUOTIENT MAPS

A Cauchy quotient map $f : (X_1, C_1) \to (X_2, C_2)$ is an onto map such that $C_2$ is the
finest Cauchy structure on $X_2$ relative to which $f$ is continuous. In other words, $f$ is a Cauchy quotient map iff \{$f(F) : F \in C_1$\} generates $C_2$.

If $f : (X_1, C_1) \to (X_2, C_2)$ is a map and there is a map $g : (X_2, C_2) \to (X_1, C_1)$ such
that $f \circ g = \text{id}_{X_2}$, then $f$ is called Cauchy retraction and $(X_2, C_2)$ is a Cauchy
retract of $(X_1, C_1)$. It is easy to see that a Cauchy retraction is a Cauchy
quotient map, and that $g$ is a Cauchy embedding of $(X_2, C_2)$ into $(X_1, C_1)$. Thus a
Cauchy retract is both a quotient space and a subspace of the domain space.

If $f : (X_1, C_1) \to (X_2, C_2)$ is an onto map with the property that $G \in C_2$ implies
$f^{-1}(G) \in C_1$, then $f$ is called a Cauchy initial map; in this case $C_1$ is called the
initial Cauchy structure on $X_1$ determined by $f$ and $(X_2, C_2)$. If $f : (X_1, C_1) \to (X_2, C_2)$
is an initial Cauchy map, let $x_y \in f^{-1}(y)$ be chosen arbitrarily for each $y \in X_2$, and
define $g : (X_2, C_2) \to (X_1, C_1)$ by $g(y) = x_y$. Clearly $g$ is a map, and $f \circ g = \text{id}_{X_2}$. Thus
each Cauchy initial map is a Cauchy retraction.

PROPOSITION 2.1. The image of a complete Cauchy space under a Cauchy quotient
map is complete.

PROOF. Let $f : (X, C) \to (Y, D)$ be a Cauchy quotient map, and $(X, C)$ a complete
Cauchy space. Let $G \in D$; then there are $F_1, \ldots, F_n \in C$ such that $f(F_1), \ldots,$
$f(F_n)$ are linked filters and $G \geq \bigcup_{i=1}^{n} f(F_i)$. Since $(X, C)$ is complete, there is
$x_i \in X_i$ such that $F_i \cap x_i \in C$ for $i = 1, \ldots, n$. Then $G \cap f(x_i) \in D$ for $i = 1, \ldots, n$,
and it follows that $(Y, D)$ is complete. 

PROPOSITION 2.2. Let $f : (X_1, C_1) \to (X_2, C_2)$ be a Cauchy retraction map. If
$(X_1, C_1)$ is regular, weakly regular, or $T_2$, then $(X_2, C_2)$ has the same property.
For any ordinal number $\beta$, $f : (X_1, r_\beta C_1) \to (X_2, r_\beta C_2)$ and $f : (X_1, w_\beta C_1) \to (X_2, w_\beta C_2)$
are Cauchy retraction maps.

PROOF. The first assertion follows from the fact that $(X_2, C_2)$ is Cauchy-
homeomorphic to a subspace of $(X_1, C_1)$. The second is obtained by applying
Proposition 1.4 to both $f$ and the associated map $g : (X_2, C_2) \to (X_1, C_1)$. 

THE REGULARITY SERIES OF A CAUCHY SPACE

PROPOSITION 2.3. If \( f : (X_1, \mathcal{C}_1) \rightarrow (X_2, \mathcal{C}_2) \) is a Cauchy initial map, then
\[ f : (X_1, r_\beta \mathcal{C}_1) \rightarrow (X_2, r_\beta \mathcal{C}_2) \] and \( f : (X_1, \nu_\beta \mathcal{C}_1) \rightarrow (X_2, \nu_\beta \mathcal{C}_2) \) are Cauchy initial maps for all ordinal numbers \( \beta \).

PROOF. First note that a Cauchy initial map is a proper map (see [7]) between the associated convergence spaces (which we denote by \((X_1, q_1)\) and \((X_2, q_2)\)); this implies that \( f(\text{cl}^n_{q_1} A) = \text{cl}^n_{q_2} f(A) \) for all \( n \in \mathbb{N} \) and \( A \subseteq X_1 \). Assume that
\[ \mathcal{G} = \text{cl}_{q_2}^m \mathcal{G}_1 \cap \ldots \cap \text{cl}_{q_2}^m \mathcal{G}_n \in r_1 \mathcal{C}_2, \] where \( \mathcal{G}_1, \ldots, \mathcal{G}_n \in \mathcal{C}_2 \) and \( \text{cl}_{q_2}^m \mathcal{G}_1, \ldots, \text{cl}_{q_2}^m \mathcal{G}_n \) are linked. It follows that \( f^{-1}(\text{cl}_{q_2}^m \mathcal{G}_i) = \text{cl}_{q_1}^m f^{-1}(\mathcal{G}_i) \) for \( i = 1, \ldots, n \) and \( \text{cl}_{q_1}^m f^{-1}(\mathcal{G}_1), \ldots, \text{cl}_{q_1}^m f^{-1}(\mathcal{G}_n) \) are also linked. Thus \( f^{-1}(\mathcal{G}) = \text{cl}_{q_1}^m f^{-1}(\mathcal{G}_1) \cap \ldots \cap \text{cl}_{q_1}^m f^{-1}(\mathcal{G}_n) \in r_1 \mathcal{C}_1 \). This establishes the result for \( \beta = 1 \). The argument extends easily by induction to an arbitrary ordinal number \( \beta \). \( \square \)

3. THE T_2 AND T_3 MODIFICATIONS.

Let \((X, \mathcal{C})\) be a weakly regular Cauchy space, and define the equivalence relation \( x \sim y \) iff \( \mathcal{C} \ni x \cap \mathcal{C} \ni y \). Let \( [x] = \{y \cap X : x \cap y \in \mathcal{C}\} \), and let \( X^\sim = \{[x] : x \in X\} \). Let \( \psi : X \rightarrow X^\sim \) be the canonical map defined by \( \psi(x) = [x] \), and let \( C^\sim \) be the quotient Cauchy structure on \( X^\sim \) induced by \( \psi : (X, \mathcal{C}) \rightarrow (X^\sim, \mathcal{C}^\sim) \).

LEMMA 3.1. If \( A \subseteq X \), where \((X, \mathcal{C})\) is a weakly regular Cauchy space, then \( wcl_{q_1} \mathcal{C} A = \psi^{-1}(\psi(A)) \).

PROOF. \( Y \subseteq wcl_{q_1} \mathcal{C} A \Leftarrow \exists A \subseteq X \text{ such that } \psi^{-1}(Y) = A \subseteq \psi^{-1}(\psi(A)) \). \( \square \)

PROPOSITION 3.2. Let \((X, \mathcal{C})\) be a weakly regular Cauchy space. Then \( \psi : (X, \mathcal{C}) \rightarrow (X^\sim, \mathcal{C}^\sim) \) is a Cauchy initial map, and \((X^\sim, \mathcal{C}^\sim)\) is \( T_2 \).

PROOF. If \( F \in \mathcal{C} \), then by Lemma 3.1, \( wcl_{q_1} \mathcal{C} F = \psi^{-1}(\psi(F)) \). Since filters of the form \( \psi(F) \), \( F \in \mathcal{C} \), generate \( \mathcal{C}^\sim \) and \( wcl_{q_1} \mathcal{C} F \in \mathcal{C} \), it follows that \( \psi \) is a Cauchy initial map.

To show that \((X^\sim, \mathcal{C}^\sim)\) is \( T_2 \), suppose \( a \cap b \in \mathcal{C}^\sim \), where \( a = [\mathcal{X}] \) and \( b = [\mathcal{Y}] \). Then \( \psi^{-1}(a \cap b) = [\mathcal{X}] \cap [\mathcal{Y}] \in \mathcal{C} \). This means \( [x] \cap [y] \neq \emptyset \), which implies \( [x] = [y] = a = b \). \( \square \)

Next, let \((X, \mathcal{C})\) be an arbitrary Cauchy space. Define \((X_h, \mathcal{C}_h) = (X^\sim, (\mathcal{C}^\sim)_h) \) and \((X_r, \mathcal{C}_r) = (X^\sim, (\mathcal{C}^\sim)_r) \). By Propositions 2.3 and 3.2, it follows that the former Cauchy space is \( T_2 \) and the latter is \( T_3 \); we shall call these the \( T_2 \) and \( T_3 \) modifications, respectively, of \((X, \mathcal{C})\). Consider the following diagram (HT):

\[
\begin{array}{c}
(X, \mathcal{C}) \xrightarrow{id} (X, \nu \mathcal{C}) \xrightarrow{id} (X, r \mathcal{C}) \xrightarrow{\psi \mathcal{C}} (X_r, \mathcal{C}_r) \\
\downarrow \psi \mathcal{C} \downarrow \psi \mathcal{C} \downarrow \theta \\
(X_h, \mathcal{C}_h) \xrightarrow{id_h} (X_h, r \mathcal{C}_h) \rightarrow ((X_h)_r, (r \mathcal{C}_h)_r)
\end{array}
\]
where \( \theta : (X, \mathcal{C}) \rightarrow ((X_h), \mathcal{C}_h) \) is defined by \( \theta([x]) = \psi_{\mathcal{C}_h}(\psi_{\mathcal{W}}(x)) \).

**PROPOSITION 3.3.** The diagram (HT) is commutative. All identity functions are maps, and all other functions are Cauchy initial maps. Furthermore, \( \theta \) is a Cauchy homeomorphism.

**PROOF.** All parts of this proposition are clear except, perhaps, the following: (a) \( \psi_{\mathcal{W}} : (X, \mathcal{C}) \rightarrow (X_h, \mathcal{C}_h) \) is a Cauchy initial map; (b) \( \theta \) is a Cauchy homeomorphism.

Statement (a) is a consequence of Proposition 2.3. To prove (b), let \( a = b \) in \( X \), where \( a = \psi_{\mathcal{C}}(x), b = \psi_{\mathcal{C}}(y) \). Let \( x \cap y \in \mathcal{C} \) since \( \psi_{\mathcal{C}} \) is a Cauchy initial map; thus \( \theta(a) = \theta(b) \), and \( \theta \) is well defined. Next, let \( \theta(a) = \psi_{\mathcal{C}_h}(\psi_{\mathcal{W}}(\psi_{\mathcal{R}}^{-1}(\theta))) \in (X_h) \). \( \theta(a) = \psi_{\mathcal{C}_h}(\psi_{\mathcal{W}}(\psi_{\mathcal{R}}^{-1}(\theta))) \) since both maps involved in the latter equations are Cauchy initial maps, \( x \cap y \in \mathcal{C} \), and it follows that \( a = \psi_{\mathcal{R}}(x) = \psi_{\mathcal{R}}(y) = b \); thus \( \theta \) is one-to-one. If \( A \in \mathcal{C}_h \), then one can show by a direct argument that \( \theta(A) = \psi_{\mathcal{R}}(\psi_{\mathcal{W}}(\psi_{\mathcal{R}}^{-1}(\psi_{\mathcal{W}}(A)))) \); the latter filter is in \( (\mathcal{R}) \) since all maps involved are Cauchy initial maps. A similar argument shows that \( \theta^{-1} \) is a map, and the proof of (b) is complete.

Let \( f : (X, \mathcal{C}) \rightarrow (Y, \mathcal{D}) \) be a Cauchy map, and define \( f_h : (X_h, \mathcal{C}_h) \rightarrow (Y_h, \mathcal{D}_h) \) by \( f_h([x]) = [f(x)] \), and \( f : (X, \mathcal{C}) \rightarrow (Y, \mathcal{D}) \) by \( f : ([x]) = [f(x)] \). If \( [x] = [y] \) in \( X \), then \( x \cap y \in \mathcal{C} \), and so \( f(x \cap y) = f(x) \cap f(y) \in \mathcal{D} \) by Proposition 1.4; thus \( [f(x)] = [f(y)] \), and \( f_h \) is well-defined. A similar argument shows that \( f_h \) is well-defined.

**PROPOSITION 3.4.** If \( f : (X, \mathcal{C}) \rightarrow (Y, \mathcal{D}) \) and \( g : (Y, \mathcal{D}) \rightarrow (Z, \mathcal{E}) \) are maps, then \( f_h, g_h, f, g \) (defined in the preceding paragraph) are maps, \( (gf)_h = g_h f_h \), and \( (gf)_h = f_h g_h \).

**PROOF.** To prove that \( f_h \) is a map, let \( A \in \mathcal{C}_h \); then \( \psi_{\mathcal{W}}(\psi_{\mathcal{R}}^{-1}(A)) \in \mathcal{C}_h \), which establishes the desired result. A similar argument shows that \( f \) is a map. Finally, note that \( (g_h \circ f_h)([x]) = g_h([f(x)]), \) \( (gf)_h([x]) = (g \circ f)_h(x); \) a similar argument establishes that \( g \circ f \). \( \square \)

**PROPOSITION 3.5.** If \( f : (X, \mathcal{C}) \rightarrow (Y, \mathcal{D}) \) is a Cauchy retraction (respectively, a Cauchy initial map) the \( f \) and \( f_h \) are Cauchy retractions (respectively, Cauchy initial maps).

**PROOF.** If \( f \) is a Cauchy retraction map, then there is a map \( g : (Y, \mathcal{D}) \rightarrow (X, \mathcal{C}) \) such that \( f \circ g = \text{id}_Y \). Then \( f_h : (X_h, \mathcal{C}_h) \rightarrow (Y_h, \mathcal{D}_h) \) and \( g_h : (Y_h, \mathcal{D}_h) \rightarrow (X_h, \mathcal{C}_h) \) are both maps, and \( (f \circ g)_h = f_h \circ g_h = (\text{id})_h = \text{id}_h \), which implies that \( f_h \) is a Cauchy retraction map. A similar argument establishes that \( f \) is a Cauchy retraction map.

Next, assume that \( f \) is a Cauchy initial map.

\[
\begin{array}{ccc}
(X, \mathcal{C}) & \rightarrow & (X, \mathcal{W}) \\
\downarrow f & & \downarrow f \\
(Y, \mathcal{D}) & \rightarrow & (Y, \mathcal{W})
\end{array}
\]
Let \( A \in D_h \). By commutativity of the preceding diagram, \( f^{-1}(\psi_{\mathcal{D}}^{-1}(A)) = \psi_{\mathcal{C}}^{-1}(f_h^{-1}(A)) \). Since \( f:(X,\mathcal{C}) \to (Y,\mathcal{D}) \) and \( \psi_{\mathcal{D}} \) are Cauchy initial maps, \( f^{-1}(\psi_{\mathcal{D}}^{-1}(A)) \in \mathcal{C} \), and so \( f_h^{-1}(A) = \psi_{\mathcal{C}}^{-1}(f_h^{-1}(A)) \in \mathcal{C}_h \). A similar argument shows that \( f_\tau \) is a Cauchy initial map. □

4. **ADMISSIBLE CONVERGENCE STRUCTURES.**

A convergence space \((X,q)\) is **Cauchy-admissible** if there is a Cauchy structure \( \mathcal{C} \) on \( X \) such that \( q = q_\mathcal{C} \). "Cauchy-admissible" will be shortened to **admissible**. Admissible convergence structures have been characterized by H. Keller [3]. We denote by \( F_q(x) \) the set of all filters on \( X \) which \( q \)-converge to \( x \).

**PROPOSITION 4.1.** The following statements about a convergence space \((X,q)\) are equivalent.

1. \((X,q)\) is admissible.
2. If \( F_q(x) \cap F_q(y) \neq \emptyset \), then \( F_q(x) = F_q(y) \).
3. \( \mathcal{C}^q = \{ F \in F(X) : F \text{ q-converges} \} \) is a Cauchy structure on \( X \).
4. \( F \rightarrow x \) iff there are linked filters \( F_1 \rightarrow x_1, \ldots, F_n \rightarrow x_n \) such that \( x \in \{ x_1, \ldots, x_n \} \) and \( F \geq F_1 \cap \ldots \cap F_n \).

Given an arbitrary convergence structure \( q \) on \( X \), define \( \alpha q \) as follows: \( F \rightarrow x \) in \((X,\alpha q)\) if there are linked filters \( F_1 \rightarrow x_1, \ldots, F_n \rightarrow x_n \) in \((X,q)\) such that \( x \in \{ x_1, \ldots, x_n \} \) and \( F \geq \bigcap_{i=1}^n F_i \). It is obvious that \( \alpha q \) is the finest admissible convergence structure on \( X \) coarser than \( q \); we shall call \( \alpha q \) the **admissible modification** of \( q \). We omit the straightforward proof of the next proposition.

**PROPOSITION 4.2.** If \( f : (X,q) \rightarrow (Y,p) \) is continuous, then \( f : (X,\alpha q) \rightarrow (Y,\alpha p) \) is continuous.

It is clear that "admissible convergence space" and "complete Cauchy space" are the same mathematical notion, formulated in slightly different ways. An R-series for convergence spaces is developed in [5] and [7], and in this section we consider the relationship between this notion and the R-series for Cauchy spaces developed in Section 1 of this paper. We begin this investigation by studying the relationship between "admissibility" and "regularity".

Starting with a convergence space \((X,q)\), the R-series \( \{ r_\alpha q \} \) of \((X,q)\) is a family of convergence structures defined on the set \( X \) as follows: \( r_0 q = q; F \rightarrow x \) relative to \( r_1 q \) iff there exist \( n \in \mathbb{N} \) and \( G \rightarrow x \) relative to \( q \) such that \( F \geq c_{r_1 q}^{n} G \); for ordinal \( \alpha > 0 \), \( F \rightarrow x \) relative to \( r_\alpha q \) iff there exist \( n \in \mathbb{N} \), \( G \rightarrow x \) relative to \( q \), and \( \beta < \alpha \) such that \( F \geq c_{r_\beta q}^{n} G \).

**LEMMA 4.3.** For any convergence space \((X,q)\) and \( A \subseteq X \), \( c_{r_1 q} A \subseteq c_{r_\alpha q} A \subseteq c_{r_{1+q} A} \).

**PROOF.** The first inclusion is obvious. Let \( x \in c_{r_\alpha q} A \); then there is \( F \rightarrow x \) in \((X,\alpha q)\) such that \( A \in F \). Also, there are linked filters \( F_1 \rightarrow x_1, \ldots, F_n \rightarrow x_n \) in \((X,q)\) such that \( F \geq \bigcap_{i=1}^n F_i \) and \( x = x_k \) for some \( k \leq n \). Clearly, there is \( k, l \leq n \), such that \( F_k \vee \hat{A} \), where \( \hat{A} \) is the filter of all oversets of \( A \). Thus \( x_\ell \in c_{r_\alpha q} A \).

Because the filters \( \{ F_1 \} \) are linked, it follows that \( x_\ell \geq c_{r_{1+q} q} x_\ell \). Thus \( x_\ell \rightarrow x_k \) in \((X,\alpha q)\), which implies \( x_k \in c_{r_{1+q} q} \{ x_\ell \} \). Thus \( x_k = x \in c_{r_{1+q} q} A \). □
PROPOSITION 4.4. If \((X, q)\) is regular, then \(\sigma q\) is regular.

PROOF. Let \(F \rightarrow x\) in \((X, q)\). Let \(F_1, \ldots, F_n\) be as in the preceding proof. Then \(\text{cl}_{\sigma q} F \geq \text{cl}_{\sigma q} (\cap F_i) \geq \text{cl}_q^2 (\cap F_i) = \cap \text{cl}_q^2 F_i\), where Lemma 4.3 is used for the second inequality. Since \(q\) is regular, \(\text{cl}_q^2 F_i \rightarrow x_i\) in \((X, q)\), and so \(\text{cl}_{\sigma q} F \rightarrow x\) in \((X, q)\).

A convergence space is defined in [5] to be symmetric if it is regular and \(F \rightarrow x\) whenever \(F \rightarrow y\) and \(y \rightarrow x\).

PROPOSITION 4.5. \((X, q)\) is symmetric iff it is admissible and regular.

PROOF. A regular, admissible space is obviously symmetric. Conversely, let \(F \geq \cap F_i\), where \(F_i \rightarrow x_i\) in \((X, q)\) for \(i = 1, \ldots, n\), and the \(\{F_i\}\) are linked. Because the \(\{F_i\}\) are linked and \(q\) is regular, \(x_i \rightarrow x_j\) for \(i, j \in \{1, \ldots, n\}\). Since \(q\) is symmetric, \(F_i \rightarrow x_j\) for \(i, j \in \{1, \ldots, n\}\). Thus \(\cap F_i \rightarrow x_j\) for \(j = 1, \ldots, n\), and so \(q\) is admissible.

For a convergence space \((X, q)\), let \(rq\) denote the regular modification of \(q\) (i.e., the terminal element of the \(R\)-series of \((X, q)\)). It is clear from the preceding propositions that \(\sigma r q\) is the finest symmetric convergence structure on \(X\) coarser than \(q\) (i.e., the symmetric modification of \(q\)); in accordance with the notation of [5], we introduce the notation \(\sigma q = \sigma r q\). A symmetric series \(\{\sigma_q (\beta)\}\) for a convergence space \((X, q)\) can be constructed as follows:

\[
\begin{align*}
\sigma_0 q &= q \\
\sigma_1 q &= \sigma r q \\
\vdots \\
\sigma_{\beta} q &= \sigma(\sigma_{\beta-1} q), \text{ if } \beta \text{ is a non-limit ordinal} \\
\sigma_{\beta} q &= \inf(\sigma_{\gamma} q : \gamma < \beta), \text{ if } \beta \text{ is a limit ordinal.}
\end{align*}
\]

If \((X, q)\) is an admissible convergence space, then we can identify \(q\) with the complete Cauchy structure \(C^q = \{F : F \in F(X) : F \text{-q-converges}\}\). A comparison of the respective definitions leads to the following result.

PROPOSITION 4.6. If \((X, q)\) is an admissible convergence space, then \(\sigma_{\beta} q = r_{\beta} C^q\) for all ordinal numbers \(\beta\). In particular, \(\sigma q = r C^q\).

Starting with an arbitrary convergence space \((X, q)\), we define the \(T_2\)-modification \((X_h, q_h) = (X_h, (C^{q_h})_h)\) and the \(T_3\)-modification \((X_t, (C^{q_t})_t)\). The latter notion was previously discussed in [7]; the former is apparently new. Given a continuous function \(f : (X, q) \rightarrow (Y, p)\), let \(f_h : (X_h, q_h) \rightarrow (Y_h, p_h)\) and \(f_t : (X_t, q_t) \rightarrow (Y_t, p_t)\) be defined as in the paragraph preceding Proposition 3.4. The next proposition is clear.

PROPOSITION 4.7. If \(f : (X, q) \rightarrow (Y, p)\) is a continuous function, then \(f_t : (X_t, q_t) \rightarrow (Y_t, p_t)\) and \(f_h : (X_h, q_h) \rightarrow (Y_h, p_h)\) are also continuous.
5. THE WYLER MODIFICATION AND COMPLETION

Let \((X, \mathcal{C})\) be a Cauchy space. If \(\mathcal{F}, \mathcal{G}\) are filters in \(\mathcal{C}\) such that \(\mathcal{F} \cap \mathcal{G} \in \mathcal{C}\), then \(\mathcal{F}\) and \(\mathcal{G}\) are defined to be \(\mathcal{C}\)-equivalent. Let \(X^* = \langle \mathcal{F} >; \mathcal{F} \in \mathcal{C} \rangle\) be the set of all \(\mathcal{C}\)-equivalence classes, and let \(j : X \to X^*\) be the canonical function \(j(x) = \langle \cdot \rangle\). Let \(\mathcal{C}^*\) be the Cauchy structure on \(X^*\) generated by \(\{j(\mathcal{F}) \cap \langle \cdot \rangle; \mathcal{F} \in \mathcal{C}\}\). We shall call \((X^*, \mathcal{C}^*)\) the Wyler modification of \((X, \mathcal{C})\).

**PROPOSITION 5.1.** (a) For any Cauchy space \((X, \mathcal{C})\), \((X^*, \mathcal{C}^*)\) is a complete Cauchy space, \(j : (X, \mathcal{C}) \to (X^*, \mathcal{C}^*)\) is a map, and \(j(X)\) is dense in \(X^*\).

(b) If \((X, \mathcal{C})\) is weakly regular, then \((X^*, \mathcal{C}^*)\) is \(T_2\).

**PROOF.** All parts of (a) follow directly from the definition of \(\mathcal{C}^*\). To prove (b), assume \(\mathcal{A} \cap \mathcal{A} \cap \mathcal{B} \in \mathcal{C}^*\). \(\mathcal{A} \in \mathcal{C}^*\) implies there are filters \(\mathcal{F}_1, \ldots, \mathcal{F}_n\) in \(\mathcal{C}\) such that \(\mathcal{F}_1 \cap \langle \mathcal{F}_1 \rangle, \ldots, \mathcal{F}_n \cap \langle \mathcal{F}_n \rangle\) are linked, \(\mathcal{A} \supseteq \cap \{j(\mathcal{F}_i) \cap \langle \mathcal{F}_i \rangle; i = 1, \ldots, n\}\), and \(a, b \in \{\langle \mathcal{F}_i \rangle; i = 1, \ldots, n\}\). It follows that \(\mathcal{F}_1 \cap \langle \mathcal{F}_1 \rangle, \ldots, \mathcal{F}_n \cap \langle \mathcal{F}_n \rangle\) are linked in \(X\). But \(\langle \mathcal{F}_1 \rangle = \mathcal{F}_1\) for \(i = 1, \ldots, n\); since \((X, \mathcal{C})\) is weakly regular, it follows that \(\langle \mathcal{F}_1 \rangle = \ldots = \langle \mathcal{F}_n \rangle = a = b\), and hence \((X^*, \mathcal{C}^*)\) is \(T_2\).

For a weakly regular Cauchy space \((X, \mathcal{C})\), let \(\psi : (X, \mathcal{C}) \to (X^*, \mathcal{C}^*)\) be the quotient map defined at the beginning of Section 3. Define \(j^\wedge : (X^*, \mathcal{C}^*) \to (X^*, \mathcal{C}^*)\) by \(j^\wedge([x]) = \langle \cdot \rangle\). Note that \([x] = [y] \Rightarrow \langle x \rangle \cap \langle y \rangle \in \mathcal{C} \Rightarrow \langle x \rangle = \langle y \rangle\); thus \(j^\wedge\) is one-to-one. Also, \(\mathcal{F} \in \mathcal{C}\) implies \(j^\wedge(\psi(\mathcal{F})) = j(\mathcal{F})\). A comparison of the definitions of \(j^\wedge\) and \(\mathcal{C}^*\) leads immediately to the following result.

**PROPOSITION 5.2.** For a weakly regular Cauchy space \((X, \mathcal{C})\), the following diagram is commutative, and \(j^\wedge\) is a dense Cauchy embedding.

\[
\begin{array}{ccc}
(X, \mathcal{C}) & \to & (X^*, \mathcal{C}^*) \\
\downarrow \psi & & \downarrow j \\
(X^*, \mathcal{C}^*) & \to & (X^*, \mathcal{C}^*)
\end{array}
\]

For a weakly regular Cauchy space \((X, \mathcal{C})\), the quotient space \((X^*, \mathcal{C}^*)\) is the \(T_2\) modification \((X_{h', \mathcal{C}_{h'}})\) of \((X, \mathcal{C})\) defined in Section 3. Thus we obtain:

**COROLLARY 5.3.** If \((X, \mathcal{C})\) is a weakly regular Cauchy space, then \((X^*, \mathcal{C}^*), j^\wedge\) is a \(T_2\) completion of \((X_{h', \mathcal{C}_{h'}})\).

In case \((X, \mathcal{C})\) is \(T_2\), \((X, \mathcal{C})\) coincides with \((X_{h', \mathcal{C}_{h'}})\) and \(j^\wedge\) with \(j\); in this case \((X^*, \mathcal{C}^*), j\) is called the Wyler completion of \((X, \mathcal{C})\). The Wyler completion is the finest in standard form (See [2], [6]). Any \(T_2\) completion \((X_1, \mathcal{C}_1)\) of \((X, \mathcal{C})\) such that any map \(f\) from \((X, \mathcal{C})\) to a complete space \((Y, \mathcal{D})\) can be lifted to map \(f_1 : (X_1, \mathcal{C}_1) \to (Y, \mathcal{D})\) such that \(f_1 \cdot h = f\) is then equivalent to the Wyler completion.

Next we examine the extension properties of the Wyler modification. If \(f : (X, \mathcal{C}) \to (Y, \mathcal{D})\) is a map, define \(f^* : (X^*, \mathcal{C}^*) \to (Y^*, \mathcal{D})\) by \(f^*(\langle \cdot \rangle) = \langle f(\cdot) \rangle\).

If \(\langle \mathcal{F} \rangle \in \mathcal{C}\), then \(\mathcal{F} \in \mathcal{D}\) implies \(f(\mathcal{F}) \in \mathcal{D}\); thus \(\langle f(\mathcal{F}) \rangle = \langle f(\mathcal{F}) \rangle\) is well defined. Also, one easily verifies that if \(f : (X, \mathcal{C}) \to (Y, \mathcal{D})\) and \(g : (Y, \mathcal{D}) \to (Z, \mathcal{B})\) are maps, then \((g \cdot f)^* = g^* \cdot f^* : (X^*, \mathcal{C}^*) \to (Z^*, \mathcal{B}^*)\).
PROPOSITION 5.4. If \((X, \mathcal{C})\) and \((Y, \mathcal{D})\) are Cauchy spaces and \(f : (X, \mathcal{C}) \to (Y, \mathcal{D})\) is a map, Cauchy quotient map, Cauchy retraction, or Cauchy initial map, then \(f^* : (X^*, \mathcal{C}^*) \to (Y^*, \mathcal{D}^*)\) has the same property.

PROOF. Consider the commutative diagram
\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow j_X \quad \downarrow j_Y \\
(\mathcal{C}^*) \xrightarrow{f^*} (\mathcal{D}^*).
\end{array}
\]
If \(j_X(F) \cap \mathcal{F}_C^* \) is a generating element for \(\mathcal{C}^*\), then \(f^*(j_X(F) \cap \mathcal{F}_C^*) = j_Y(f(F) \cap \mathcal{F}_D^*)\) is in \(\mathcal{D}^*\), and so \(f^*\) is a map. If \(f\) is a Cauchy quotient map and \(\mathcal{F}_D^* \in Y^*\), then \(\mathcal{G} \in \mathcal{D}\) implies there are \(F_1, \ldots, F_n \in \mathcal{C}\) such that \(f(F_1), \ldots, f(F_n)\) are linked and \(\mathcal{G} \supseteq \bigcap \{f(F_i) : i = 1, \ldots, n\}\). Then \(f^*(\mathcal{F}_D^*) = \mathcal{G}\) for \(i = 1, \ldots, n\), and \(f^*\) is onto \(\mathcal{D}^*\). Also, if \(A = j_Y(\mathcal{G}) \cap \mathcal{F}_D^*\) is a generating filter in \(\mathcal{D}^*\), where \(\mathcal{G} \in \mathcal{D}\) and \(F_1, \ldots, F_n\) are as described above, then \(A \supseteq f^*(\bigcap j_X(F_i) \cap \mathcal{F}_C^* : i = 1, \ldots, n)\); from this it follows that \(f^*\) is a Cauchy quotient map.

Next assume that \(f : (X, \mathcal{C}) \to (Y, \mathcal{D})\) is a Cauchy retraction; then there is a map \(g : (Y, \mathcal{D}) \to (X, \mathcal{C})\) such that \(f \circ g = \text{id}_Y\). By the remarks preceding the proposition, \((f \circ g)^* = (\text{id}_Y)^* = \text{id}_{Y^*} : (Y^*, \mathcal{D}^*) \to (Y^*, \mathcal{D}^*)\) and it follows that \(f^* : (X^*, \mathcal{C}^*) \to (Y^*, \mathcal{D}^*)\) is a retraction map.

Finally, assume that \(f : (X, \mathcal{C}) \to (Y, \mathcal{D})\) is a Cauchy initial map, and let \(A = j_Y(\mathcal{G}) \cap \mathcal{F}_D^*\) be a generating element of \(\mathcal{D}^*\), where \(\mathcal{G} \in \mathcal{D}\). By a direct argument, one can show that \((f^*)^{-1}[(j_Y(\mathcal{G}) \cap \mathcal{F}_D^*)] \supseteq j_X f^{-1}(\mathcal{G}) \cap \mathcal{F}_C^*\), and so \(f^*\) is also a Cauchy initial map. \(\square\)

PROPOSITION 5.5. If \((X, \mathcal{C})\) is a weakly regular Cauchy space, then \(((X^*, \mathcal{C}^*), j^*)\) is the Wyler completion of \((X_h, \mathcal{C}_h)\).

PROOF. Let \(f : (X_h, \mathcal{C}_h) \to (Y, \mathcal{D})\) be a map, where \((Y, \mathcal{D})\) is complete and \(T_2\), and let \(\psi : (X, \mathcal{C}) \to (X_h, \mathcal{C}_h)\) be the canonical quotient map.

\[
\begin{array}{c}
(X, \mathcal{C}) \xrightarrow{f} (X^*, \mathcal{C}^*) \\
\downarrow \quad \downarrow (f \circ \psi)^* \\
(X_h, \mathcal{C}_h) \xrightarrow{\psi} (Y, \mathcal{D})
\end{array}
\]
By Corollary 5.3, \(((X^*, \mathcal{C}^*), j^*)\) is a \(T_2\) completion of \((X_h, \mathcal{C}_h)\), and by Proposition 5.4, the map \(f \circ \psi\) has a continuous extension \((f \circ \psi)^* : (X^*, \mathcal{C}^*) \to (Y^*, \mathcal{D}^*) = (Y, \mathcal{D})\). Since the above diagram is commutative, the completion \(((X^*, \mathcal{C}^*), j^*)\) has the lifting property which characterizes the Wyler completion of \((X_h, \mathcal{C}_h)\). \(\square\)

The next proposition follows from Propositions 1.4, 2.2, 2.3, and 5.4.

PROPOSITION 5.6. Let \(f : (X, \mathcal{C}) \to (Y, \mathcal{D})\) be a map. Then \(f^* : (X^*, \mathcal{W}_B \mathcal{C}^*) \to (Y^*, \mathcal{W}_B \mathcal{D}^*)\) and \(f^* : (X^*, \mathcal{I}_B \mathcal{C}^*) \to (Y^*, \mathcal{I}_B \mathcal{D}^*)\) are maps for all ordinals \(\beta\). If \(f\) is a Cauchy retraction or Cauchy initial map, then both of the maps labeled \(f^*\) have the same property.
Let \( f : (X, \mathcal{C}) \to (Y, \mathcal{D}) \) be a map, and consider the diagram (D*):

\[
\begin{array}{cccc}
(X, \mathcal{C}) & \to & (X^*, \mathcal{C}^*) & \to & (X^*_h, \mathcal{C}^*_h) & \to & (X^*_\tau, \mathcal{C}^*_\tau) \\
f \downarrow & & \downarrow f^* & & \downarrow (f^*)_h & & \downarrow (f^*_\tau) \\
(Y, \mathcal{D}) & \to & (Y^*, \mathcal{D}^*) & \to & (Y^*_h, \mathcal{D}^*_h) & \to & (Y^*_\tau, \mathcal{D}^*_\tau)
\end{array}
\]

where the unlabeled horizontal arrows are the canonical quotient maps. The next proposition is an immediate consequence of Proposition 3.5 and 5.4.

**Proposition 5.7.** In diagram (D*), if \( f \) is a map (respectively, Cauchy retraction, Cauchy initial map) then each vertical arrow in the diagram is a map (respectively, Cauchy retraction, Cauchy initial map).

6. **T_3 completions.**

In [6], a Cauchy space which has a T_3 completion is defined to be a C_3 Cauchy space. In this section a characterization of C_3 spaces is given which makes use of the R-series along with a set function \( \Xi \) defined originally in [4]. The main theorem (Theorem 6.4) is due to G. D. Richardson.

Throughout this section, \((X, \mathcal{C})\) is assumed to be a T_3 Cauchy space. The Wyler completion of \((X, \mathcal{C})\) will be denoted by \(((X^*, \mathcal{C}^*), j)\). Let \( p \) be the convergence structure on \( X^* \) compatible with \( \mathcal{C}^* \), and let \( p_\alpha \) denote the convergence structure on \( X^* \) compatible with \( r_\alpha p \) (see Section 4). As usual, \( N \) denotes the set of natural numbers; also recall that \( \hat{\alpha} \) denotes the filter consisting of all oversets of \( \alpha \).

We next construct a family of set functions \( \Sigma^n_\alpha \) which, for each \( n \in N \) and each ordinal number \( \alpha \), map subsets of \( X \) into subsets of \( X^* \). If \( F \) is a filter on \( X \), then \( \Sigma^n_\alpha(F) \) is defined to be the filter on \( X^* \) generated by \( \{ \Sigma^n_\alpha(F) : F \in F \} \). For any subset \( A \) of \( X \) and \( n \in N \), we define:

\[
\Sigma_0^n(A) = \{ <F> \in X^* : \text{there is } G \in <F> \text{ such that } j(G) \cup j(\hat{A}) \}
\]

\[
\vdots
\]

\[
\Sigma_0^n(A) = \{ <F> \in X^* : \text{there is } G \in <F> \text{ such that } j(G) \cup \Sigma^{n-1}_0(\hat{A}) \}.
\]

If \( \alpha \) is an ordinal number, \( A \subseteq X \), and \( n \in N \), we define:

\[
\Sigma^n_\alpha(A) = \{ <F> \in X^* : \text{there is } \beta < \alpha, k \in N, \text{ and } G < F \text{ such that } \Sigma^k_\beta(G) \cup j(\hat{A}) \}
\]

\[
\vdots
\]

\[
\Sigma^n_\alpha(A) = \{ <F> \in X^* : \text{there is } \beta < \alpha, k \in N, \text{ and } G < F \text{ such that } \Sigma^k_\beta(G) \cup \Sigma^{n-1}_\alpha(\hat{A}) \}.
\]

**Proposition 6.1.** For all \( n \in N \) and \( A \subseteq X \), \( \Sigma_0^n(A) = cl_p^n j(A) \).

**Proof.** Assume that the equality holds for \( n \). If \( <F> \in \Sigma_0^{n+1}(A) \), then there is \( G < F \) such that \( j(G) \cup \Sigma_0^n(A) \) exists. Using the induction assumption and the fact that \( j(G) \) \( p \)-converges to \(<F>\), it follows that \( <F> \in cl_p^{n+1} j(A) \). Conversely, if \( <F> \in cl_p^{n+1} j(A) \), then there is a filter \( \downarrow p \)-converging to \(<F>\) such that \( cl_p^n j(A) \).
By construction of \( C^* \), \( \Phi \geq j(6) \) for some \( 6 \in \langle C \rangle \), and by the induction assumption, 
\( j(6) \lor P(A) \) exists. Thus \( \langle F \rangle \in \Sigma_{P}^{n+1}(A) \). For \( n = 1 \), the preceding argument can be applied if we define \( \Sigma_{P}^{0}(A) = \text{cl}_{P}^{1}(A) = j(A) \).

PROPOSITION 6.2. For all ordinal numbers \( \alpha \), for all \( n \in \mathbb{N} \), and for all subsets \( A \) of \( X \), \( \Sigma_{P}^{n}(A) = \text{cl}_{P}^{n}(A) = j(A) \).

PROOF. Consider all pairs \( P \) of the form \( (\alpha, n) \), where \( \alpha \) is an ordinal number and \( n \in \mathbb{N} \); let \( P \) be ordered as follows: \( (\beta, m) < (\alpha, n) \) if \( \beta < \alpha \) or \( \beta = \alpha \) and \( m < n \). Since \( P \) is obviously well-ordered, the proof will proceed by induction.

Assume that the above equality holds for \( (\beta, m) < (\alpha, n) \); in view of the preceding proposition, we may assume \( \alpha > 1 \). If \( n > 1 \), then the induction assumption states that \( \Sigma_{P}^{n-1}(A) = \text{cl}_{P}^{n-1}(A) = \Sigma_{P}^{k}(A) = \text{cl}_{P}^{k}(A) \) for all \( \beta < \alpha \) and \( k \in \mathbb{N} \); using these equalities, the argument used in Proposition 6.1 can be repeated to establish the desired result in this case.

Finally, assume \( n = 1 \) and let \( \langle F \rangle \in \Sigma_{P}^{1}(A) \). Then there is \( 6 \in \langle F \rangle \), \( k \in \mathbb{N} \), and \( \beta < \alpha \) such that \( \Sigma_{P}^{k}(6) \lor j(A) \). Since \( \Sigma_{P}^{k}(6) \) \( \text{P}_{\alpha} \)-converges to \( \langle F \rangle \), it follows that \( \langle F \rangle \in \text{cl}_{P}^{1}(A) \). Conversely, if \( \langle F \rangle \in \text{cl}_{P}^{1}(A) \), then some ultrafilter \( \Phi \) on \( X^* \) containing \( j(A) \) \( \text{P}_{\alpha} \)-converges to \( \langle F \rangle \). By the definition of \( P \) and \( \text{P}_{\alpha} \) (see Section 4), there is \( 6 \in \langle F \rangle \), \( \beta < \alpha \), and \( k \in \mathbb{N} \) such that \( \Phi \geq \text{cl}_{P}^{k}(6) \). Employing the induction hypothesis, we have \( \Sigma_{P}^{k}(6) \lor j(A) \), and hence \( \langle F \rangle \in \Sigma_{P}^{1}(A) \). This completes the proof.

Now let \((X, \mathcal{C})\) be a \( T_3 \) Cauchy space, and consider the following two properties:

(\( P_1 \)) If \( F, 6 \in \mathcal{C} \) and, for each ordinal \( \alpha \) and \( n \in \mathbb{N} \), \( \Sigma_{P}^{n}(F) \lor \Sigma_{P}^{n}(6) \), then \( F \cap 6 \in \mathcal{C} \).

(\( P_2 \)) For each ordinal \( \alpha, n \in \mathbb{N} \), and \( F \in \mathcal{C} \), \( j^{-1}(\Sigma_{P}^{n}(F)) \in \mathcal{C} \).

PROPOSITION 6.3. If \((X, \mathcal{C})\) is a \( T_3 \) Cauchy space which satisfies condition \( (P_1) \), then for each ordinal number \( \alpha \), \( r_{\alpha} \mathcal{C}^* \) is \( T_2 \), and hence compatible with \( \text{P}_{\alpha} = r_{\alpha} \).

PROOF. The fact that \( \text{P}_{\alpha} = T_2 \) is an immediate consequence of Proposition 6.2 and property \( (P_1) \). Since a \( T_2 \) convergence structure is admissible and hence symmetric, \( \{ \text{P}_{\alpha} \} \) coincides with the symmetric series \( \{ r_{\alpha} \} \), which in turn coincides with the Cauchy R-series \( (r_{\alpha} \mathcal{C}^*) \) by Proposition 4.6.

THEOREM 6.4. The following statements about a \( T_3 \) Cauchy space \((X, \mathcal{C})\) are equivalent:

(1) \((X, \mathcal{C})\) is a \( C_3 \) Cauchy space.
(2) \((X, \mathcal{C})\) satisfies conditions \((P_1)\) and \((P_2)\).
(3) \((X^*, r\mathcal{C}^*), j)\) is a \( T_3 \) completion of \((X, \mathcal{C})\).

PROOF. The equivalence of (1) and (3) are well known (see [2] or [6]). If \((X, \mathcal{C})\) satisfies the two conditions, then the set \( D = \{ \Phi \in \mathcal{F}(X^*) : \text{there is ordinal} \ \alpha, n \in \mathbb{N}, \text{and} \ \mathcal{F} \in \mathcal{C} \text{ such that} \ \Phi \geq \Sigma_{P}^{n}(\mathcal{F}) \text{ is easily seen to be a complete} \ T_3 \text{ Cauchy structure on} \ X^* \}; \text{using Propositions 6.2 and 6.3, it follows easily that} \ D = r\mathcal{C}^* \). Condition \((P_2)\) guarantees that \( j : (X, \mathcal{C}) \rightarrow (X^*, r\mathcal{C}^*) \) is a Cauchy embedding. Thus (2) \( = \) (3). Conversely, if (3) is assumed, then \((P_1)\) follows from Proposition 6.2.
and the fact that $rC^*$ is $T_2$, while $(P_2)$ follows because the $j: (X, C) \rightarrow (X^*, rC^*)$ is a Cauchy embedding. □

REFERENCES