SOME INVARIANT THEOREMS ON GEOMETRY OF EINSTEIN NON-SYMMETRIC FIELD THEORY

LIU SHU-LIN
Institute of Mathematics
The Academy of Sciences of China

XU SEN-LIN
Department of Mathematics
University of Science and Technology of China
and
Department of Mathematics
Princeton University, New Jersey 08544
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ABSTRACT. This paper generalizes Einstein's theorem. It is shown that under the transformation

\[ T_\Lambda : U^\Lambda_{ik} \rightarrow \bar{U}^\Lambda_{ik} = U^\Lambda_{ik} + \Lambda^\Lambda_{ik}, \]

curvature tensor \( S^i_{km}(U) \), Ricci tensor \( S^i_{ik}(U) \), and scalar curvature \( S(U) \) are all invariant, where \( \Lambda = \Lambda_j dx^j \) is a closed 1-differential form on an \( n \)-dimensional manifold \( M \).

It is still shown that for arbitrary \( U \), the transformation that makes curvature tensor \( S^i_{km}(U) \) (or Ricci tensor \( S^i_{ik}(U) \)) invariant

\[ T_V : U^\Lambda_{ik} \rightarrow \bar{U}^\Lambda_{ik} = U^\Lambda_{ik} + V^\Lambda_{ik} \]

must be \( T_\Lambda \) transformation, where \( V \) (its components are \( V^\Lambda_{ik} \)) is a second order differentiable covariant tensor field with vector value.

KEY WORDS AND PHRASES. Einstein non-symmetric field, Einstein theorem, curvature tensor, Ricci tensor, scalar curvature, \( T_\Lambda \) transformation, \( T_V \) transformation.

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1. INTRODUCTION.

When A. Einstein devoted himself to research on relativism in his symmetric field \([1]\), he regarded non-symmetric \( g_{ij} \) (or \( g^{ij} \)) and non-symmetric affine connection \( D \) (its coefficients are \( \Gamma^k_{ij} \) in local coordinates \( \{x^i\} \)) as independent variables such that the number of independent variables increased from 50 (\( g_{ij} \) and \( \Gamma^k_{ij} \) are all symmetric for lower coordinates) to 80 (16 \( g_{ij} \) or \( g^{ij} \) and 64 \( \Gamma^k_{ij} \)). With so many covariant variables, it was impossible to choose them according to the
principle of relativism alone. To overcome this difficulty, Einstein introduced a very important concept, transposition invariance. This "transposition invariance" (or transposition symmetry) meant that when all $A_{ik}$ were transposed ($A_{ik}^T = A_{ki}$), all equations were still applicable [2]. Einstein supposed that field equations were transposition invariant. He thought that in physics this hypothesis was equivalent to the law that positive and negative electricity occurred symmetrically.

As the Ricci tensor $R_{ik}(F)$ represented by connected coefficients $\Gamma^{k}_{ik}$ was not transposition invariant, Einstein introduced a "pseudo-tensor" $U^k_{ik}$ instead [3]; its definition was

$$U^k_{ik} \equiv \Gamma^k_{ik} - \Gamma^l_{it} \delta^k_l, \quad \text{where} \quad \Gamma^l_{it} = \frac{\partial \Gamma^l_{it}}{\partial x^t}. \quad (1.1)$$

Denoting $\Gamma^{k}_{ik}$ by $U^k_{ik}$, we obtained

$$\Gamma^{k}_{ik} = U^k_{ik} - \frac{1}{3} U^t_{it} \delta^k_t \quad (i, k, \ell = 1, \ldots, 4). \quad (1.2)$$

Then the Ricci curvature denoted by $U$ was

$$R_{ik} = U^k_{is} - U^t_{it} U^s_{ik} + \frac{1}{3} U^s_{is} U^t_{tk} = \frac{S_{ik}(U)}{S_{ik}}. \quad (1.3)$$

Einstein proved that $S_{ik}$ were transposition invariant and the following.

**THEOREM (EINSTEIN).** [1] Under the transformation

$$T_{\lambda} : U^k_{ik} \rightarrow U^k_{ik} = U^k_{ik} + \frac{\partial \Gamma^{k}_{ik}}{\partial x^\lambda} \quad (1.4)$$

the Ricci tensor $S_{ik}$ of $U$ is invariant; i.e., under the transformation (1.4), there are $S_{ik} = S_{ik}$ for arbitrary $U$, where $S_{ik} = S_{ik}(U)$. In (4), $\lambda, \ell = 1, \ldots, 4$ and $\lambda$ is a differentiable function on a manifold $M$.

**REMARK.** Einstein gave transformation (1.4) for $n=4$, but we will still call transformation (1.4) the Einstein transformation for general $n (\geq 2)$.

One asks naturally, how about the converse of Einstein's theorem? A. Einstein and B. Kaufman did not solve the problem. It has remained unsolved.

In this paper, we generalize Einstein and Kaufman's results to an arbitrary $n (\geq 2)$ dimensional manifold $M$. Objects which we discuss are not limited to the Ricci tensor $S_{ik}$ of $U$. Besides $S_{ik}$, we discuss curvature tensor $S_{km}$ and scalar curvature $S$.

Then, for general $n (\geq 2)$, we give some invariant theorems on curvature tensor $S_{km}$, Ricci tensor $S_{ik}$, and scalar curvature $S$ of $U$. For this, first we generalize Einstein's transformation. Finally, we give converse theorems of theorems for arbitrary $n (\geq 2)$. These are the main results of this paper. In the special case $n=4$, we answer the problem above mentioned; that is, a converse to Einstein's theorem.

2. DEFINITION AND MAIN RESULTS.

To give the definitions for curvature tensor $S_{km}$ and Ricci tensor $S_{ik}$ of $U$, first let us give reasonable definitions for curvature tensor $R_{km}$ and Ricci tensor $R_{ik}$ of connection $D (\Gamma_{ik}^k)$ (order of lower coordinates is very important; what we give here differs by a minus sign from what is sometimes used, for example, in Pauli's relativism).

$$R_{km} = \Gamma_{km}^i - \Gamma_{km}^i \Gamma_{i\ell}^s - \Gamma_{km}^i S_{i\ell}^s + \Gamma_{km}^i S_{i\ell}^s$$

$$= (\Gamma_{km}^i - \Gamma_{km}^i \Gamma_{i\ell}^s) - (\Gamma_{km}^i \Gamma_{i\ell}^s)$$

$$\equiv \left[ \frac{\delta}{\partial m}, \frac{\delta}{\partial n} \right]. \quad (2.1)$$
In (2.1), let \( i = m \). Adding from 1 to \( n \) the curvature tensor \( R_{k \ell m}^i \) is contracted to obtain the Ricci tensor

\[
R_k^\ell = R_{k \ell s}^s = R_{k \ell, s}^s - R_{k \ell s, \ell}^s + R_{k \ell r}^s r_{s r}^\ell.
\] (2.2)

To establish expressions for the curvature tensor \( S_{k \ell m}^i \), Ricci tensor \( S_{i k}^i \), and scalar curvature \( S \) of \( U \) for arbitrary \( n (\geq 2) \), it is necessary to give transformation between \( U \) and \( \Gamma \) for arbitrary \( n (\geq 2) \).

In (1.1), let \( \ell = k \), add from 1 to \( n \) obtaining

\[
- U_{\ell i t} = - (n-1) U_{i t} \quad (i, k, \ell = 1, \ldots, n) \quad (2.3)
\]

Substituting (2.3) into (1.1), we can solve

\[
r_{i k}^\ell = U_{i k}^\ell - \frac{1}{n-1} \delta_{k i}^\ell U_{i t}^t \quad (i, k, \ell = 1, \ldots, n) \quad (2.4)
\]

From (2.1) - (2.3), and definitions we obtain immediately

**PROPOSITION 1.** Curvature tensor \( S_{k \ell m}^i \), Ricci tensor \( S_{i k}^i \), and scalar curvature \( S \) of \( U \) are respectively

\[
(1) \quad R_{k \ell m}^i = U_{k \ell, m}^i - \frac{1}{n-1} \delta_{k i}^\ell U_{k t}^t U_{i t}^t + \frac{1}{n-1} \delta_{k i}^\ell U_{s t}^t U_{s k}^s - \frac{1}{n-1} \delta_{k i}^\ell U_{s t}^t U_{s k}^s + \frac{1}{n-1} \delta_{k i}^\ell U_{s t}^t U_{s k}^s - \frac{1}{n-1} \delta_{k i}^\ell U_{s t}^t U_{s k}^s.
\]

\[
(2) \quad S_{i k}^i = U_{i k, s}^i - U_{i t}^t U_{s k}^s + \frac{1}{n-1} U_{s t}^t U_{s k}^s + \frac{1}{n-1} U_{i s}^s U_{i t}^t \quad (2.5)
\]

\[
(3) \quad S = g_{i k} R_{i k}^i = g_{i k} U_{i k, s}^i - g_{i t} U_{i t}^t U_{s k}^s + \frac{1}{n-1} g_{i s} U_{i t}^t U_{s k}^s \equiv S(2.6)
\]

When \( n \geq 2 \), it is not difficult to verify that Ricci tensor \( S_{i k}^i \), and scalar curvature \( S \) of \( U \) are transposition invariant.

**THEOREM 1.** Curvature tensor \( S_{k \ell m}^i \), Ricci tensor \( S_{i k}^i \), and scalar curvature \( S \) of \( U \) are all invariant under the following transformation

\[
\Lambda: U_{i k}^i \rightarrow U_{i k}^i \equiv U_{i k}^i + \delta_{i k} \Lambda = \delta_{k i} \Lambda = \delta_{k i} \Lambda, \quad (2.8)
\]

where \( \Lambda = \Lambda_j^i \) is a closed 1-differential form on a manifold \( M \); i.e., \( d\Lambda = 0 \).

**REMARK.** The transformation (2.8) is a generalization of Einstein's transformation (1.4). In fact, as \( \Lambda \) is a closed 1-differential form on a manifold \( M \) (\( d\Lambda = 0 \)), then by the Poincaré Lemma, there exists a coordinate neighborhood \( M_\epsilon \subset M \) and a differentiable function \( \lambda \) such that \( \Lambda_k = \frac{\partial \lambda}{\partial x^k} = \lambda \) (\( k = 1, \ldots, n \)). Therefore, in a local neighborhood, for example \( M_1 \), the transformation (2.8) conforms with Einstein's transformation (1.4).
Because an exact differential form $d\lambda$ is a closed differential form ($d^2 \lambda = 0$), $T_\lambda$ is a transformation which makes $S^i_{k\ell m}, S^i_{1k}$ and $S$ invariant. When $n=4$, Einstein's theorem is a special case of the above theorem 1.

**PROOF.** In local coordinates $(x^i)$, let $A_i = \lambda, \eta$, then

$$S^i_{k\ell m} = S^i_{k\ell m}(U)$$

$$= (U^i \delta^i_{kt,m} + \delta^i_{k\ell,m} - \delta^i_{k\ell,km}) - \frac{1}{n-1} \delta^i_{k\ell,m} (U^t_{kt,m} + \delta^t_{kt,km} - \delta^t_{k\ell,km})$$

$$- \frac{1}{n-1} \delta^i_{k\ell,m} (U^t_{st} + \delta^t_{st,m} - \delta^t_{s\ell,m}) (U^i_{km} + \delta^i_{km,s} - \delta^i_{s\ell,k})$$

$$+ \frac{1}{n-1} \delta^i_{s\ell,m} (U^t_{st} + \delta^t_{st,m} - \delta^t_{s\ell,m}) (U^i_{ks} + \delta^i_{ks,s} - \delta^i_{s\ell,k})$$

$$= \frac{1}{n-1} \delta^i_{k\ell,m} (U^i_{km} + \delta^i_{km,s} - \delta^i_{s\ell,k}) + \frac{1}{n-1} \delta^i_{k\ell,m} (U^i_{kt} + \delta^i_{k\ell,t} - \delta^i_{s\ell,t})$$

$$+ \frac{1}{n-1} \delta^i_{k\ell,m} (U^i_{s\ell,t} + \delta^i_{s\ell,t} - \delta^i_{s\ell,t}) (U^i_{ks} + \delta^i_{ks,s} - \delta^i_{s\ell,k})$$

$$= S^i_{k\ell m}(U) - U^i_{k\ell,m}, + \frac{1}{n-1} \delta^i_{k\ell,m} + \frac{1}{n-1} \delta^i_{k\ell,m} - \delta^i_{k\ell,m}$$

$$+ \frac{1}{n-1} \delta^i_{k\ell,m} (U^i_{km} + \delta^i_{km,s} - \delta^i_{s\ell,k}) + \frac{1}{n-1} \delta^i_{k\ell,m} (U^i_{kt} + \delta^i_{k\ell,t} - \delta^i_{s\ell,t})$$

$$+ \frac{1}{n-1} \delta^i_{k\ell,m} (U^i_{s\ell,t} + \delta^i_{s\ell,t} - \delta^i_{s\ell,t}) (U^i_{ks} + \delta^i_{ks,s} - \delta^i_{s\ell,k})$$

$$= S^i_{k\ell m}(U) - U^i_{k\ell,m}, + \frac{1}{n-1} \delta^i_{k\ell,m} + \frac{1}{n-1} \delta^i_{k\ell,m} - \delta^i_{k\ell,m}$$

where $[m, \ell] = -\lambda, \eta U^i_{km} + \ldots - \frac{n^2}{(n-1)^2} \delta^i_{m\ell,k\ell,km}$.
Consider the transformation
\[ T_\Omega : U_{ik} \to \bar{U}_{ik} = U_{ik} + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}}, \tag{2.9} \]
where \( \Omega \) is a 1-differential form, in local coordinates \( \{x^i\} \), \( \Omega = \Omega_j dx^j \).

Having the above result for theorem 1, we ask naturally if the transformation which makes curvature tensor \( S_{k\ell m}^i \) (or Ricci tensor \( S_{ik} \)) invariant is the transformation \( T_\Omega \)? For this, although we cannot give the complete answer - it is a very difficult problem - we have the following results.

**THEOREM 2.** The transformation that makes curvature tensor \( S_{k\ell m}^i \) (or Ricci tensor \( S_{ik} \)) of some \( U \) invariant
\[ T_\Omega : U_{ik} \to \bar{U}_{ik} = U_{ik} + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \]
must be \( T_\Omega \), where \( \Omega = \Omega_j dx^j \) is a 1-differential form.

**PROOF.** Similar to the proof of theorem 1, we obtain \( S_{k\ell m}^i = S_{k\ell m}^i(\bar{U}) \)

\[
= \left( U_{ik}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right) + \frac{1}{n-1} \delta^i_{\Omega} \left( U_{kt,m}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right)
- \left( U_{sk}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right) \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^s_{\Omega} \left( U_{st}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) \left( U_{mt}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^t_{\Omega} \left( U_{mt}^i + \delta^t_{\Omega} - \delta^t_{\bar{\Omega}} \right) \left( U_{tm}^i + \delta^t_{\Omega} - \delta^t_{\bar{\Omega}} \right)
- \frac{1}{(n-1)} \delta^i_{\Omega} \left( U_{mt}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) \left( U_{km}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) - [m, \ell] \]

\[ = S_{k\ell m}^i(U) + \left( U_{k\ell m}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right) + \frac{1}{n-1} \delta^i_{\Omega} \left( U_{kt,m}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right)
- \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right) \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^s_{\Omega} \left( U_{st}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) \left( U_{mt}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^t_{\Omega} \left( U_{mt}^i + \delta^t_{\Omega} - \delta^t_{\bar{\Omega}} \right) \left( U_{tm}^i + \delta^t_{\Omega} - \delta^t_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^i_{\Omega} \left( U_{mt}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) \left( U_{km}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) - [m, \ell] \]

\[ = S_{k\ell m}^i(U) + \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right) + \frac{1}{n-1} \delta^i_{\Omega} \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right)
- \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right) \left( U_{km}^i + \delta^i_{\Omega} - \delta^i_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^s_{\Omega} \left( U_{st}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) \left( U_{mt}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^t_{\Omega} \left( U_{mt}^i + \delta^t_{\Omega} - \delta^t_{\bar{\Omega}} \right) \left( U_{tm}^i + \delta^t_{\Omega} - \delta^t_{\bar{\Omega}} \right)
- \frac{1}{n-1} \delta^i_{\Omega} \left( U_{mt}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) \left( U_{km}^i + \delta^s_{\Omega} - \delta^s_{\bar{\Omega}} \right) - [m, \ell] \]
\textbf{THEOREM 3.} A necessary and sufficient condition that transformation $T_{\omega}$ makes scalar curvature $S$ of some $U$ invariant is

$$g_{ik} \omega_{k,i} - g_{ik} \omega_{i,k} = 0.$$  

where $\omega_{k,i} = \frac{\partial \omega_k}{\partial x^i}$.  

\textbf{PROOF.} From $S_{ik} = g_{ik} S_{ik} + g_{ik} (\omega_{k,i} - \omega_{i,k}) = S + (g_{ik} \omega_{k,i} - g_{ik} \omega_{i,k})$ it follows that $S = S$ if and only if $g_{ik} \omega_{k,i} - g_{ik} \omega_{i,k} = 0$.  

\textbf{REMARK.} If $g_{ik}$ are symmetric, then $g_{ik} \omega_{k,i} - g_{ik} \omega_{i,k} = 0$. If $g_{ik}$ are not symmetric, for example $g_{12} \neq g_{21}$, let

Now we give the converse of theorem 1. For this, what we must emphasize is that because of theorem 1, the transformation $T_{\omega}$ makes curvature tensor $S_{ik}^m$ and Ricci tensor $S_{ik}$ of every $U$ invariant.

The following theorems, 4 and 5 respectively, are the converses of theorem 1 on curvature tensor $S_{ik}^m$ and Ricci tensor $S_{ik}$.  

\textbf{THEOREM 4.} Let $V$ be a second order differentiable covariant tensor field with vector value and its components be $V_{ik}^l$ in local coordinates $\{x^i\}$. If the transformation

$$T_{V} : U_{ik}^l \rightarrow U_{ik}^l = U_{ik}^l + V_{ik}^l$$

makes curvature tensor $S_{ik}^m$ of every $U$ invariant, then it implies

$$V_{ik}^l = \delta_{ik}^l \omega_{k,i} - \delta_{ik}^l \omega_{i,k},$$

where $\omega = \sum dx^j$ is a closed 1-differential form; i.e., $T_{V}$ must be $T_{\omega}$.  

\textbf{PROOF.} By (2.10),

$$S_{ik}^m = S_{ik}^m(\bar{U}) = (U_{ik,m}^t + V_{ik,m}^t) - \frac{1}{n-1} \delta_{ik}^l (U_{kt,m}^t + V_{kt,m}^t)$$

$$- (U_{ik}^t + V_{ik}^t) (U_{km}^t + V_{km}^t) + \frac{1}{n-1} \delta_{ik}^l (U_{st}^t + V_{st}^t) (U_{km}^t + V_{km}^t)$$

Therefore, $S_{ik}^m = S_{ik}^m$ if and only if

$$\omega_{k,m} = \omega_{m,k}, \text{ i.e., } d\omega = 0.$$
\[ + \frac{1}{n-1} \left( U_{kt}^t + V_{kt}^t \right) \left( U_{m}^m + V_{m}^m \right) - \frac{1}{n-1} \delta_{\ell} \left( U_{mt}^t + V_{mt}^t \right) \left( U_{ks}^s + V_{ks}^s \right) \]

- \left[ m, \ell \right] = S_{k\ell m}^i + S_{k\ell m}^i - \frac{1}{n-1} V_{kt,m}^t \delta_{\ell}^i - V_{s,k\ell m}^i - U_{k\ell m}^s \]

\[- V_{s,k\ell m}^s + \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^s \]

+ \frac{1}{n-1} V_{kt,m}^t + \frac{1}{n-1} V_{m}^t V_{kt,m}^s + \frac{1}{n-1} \delta_{\ell} U_{k\ell m}^t \]

- \left[ m, \ell \right] = S_{k\ell m}^i + \frac{1}{n-1} V_{k\ell m}^s.

Since \( S_{k\ell m}^i = S_{k\ell m}^i \) (for every \( U \)), now \( F_{k\ell m}^i \)(U) = 0 \( i, k, \ell, m = 1, \ldots, n \) and for every \( U \). Therefore,

\[ 0 = \frac{\partial F_{k\ell m}^i}{\partial U_{k\ell m}^i} = -V_{\gamma k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta - V_{k\ell m}^i \delta_{\gamma}^\alpha \delta_{\ell}^\beta + \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta \]

+ \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta + \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta - \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta - \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta \]

In the above formula, let \( i = \ell \). Adding from 1 to \( n \) we obtain

\[ 0 = \frac{\partial F_{k\ell m}^i}{\partial U_{k\ell m}^i} = -V_{\gamma k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta - V_{k\ell m}^i \delta_{\gamma}^\alpha \delta_{\ell}^\beta + \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta \]

+ \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta - \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta - \frac{1}{n-1} \delta_{\ell} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta \]

In (2.11) let \( \gamma = \gamma \) again. Adding from 1 to \( n \) we obtain

\[ 0 = n V_{\gamma m}^\ell \delta_{\ell}^\alpha \delta_{\gamma}^\beta - \frac{1}{n-1} V_{k\ell m}^i \delta_{\ell}^\alpha \delta_{\gamma}^\beta \]
nV^B_{lm} + V^\alpha_{\alpha m} - n^2 V^\beta_{lkm} - \frac{1}{n-1} V^t_{mt} = - \frac{V^\beta_{\alpha m}}{n-1} - \frac{V^t_{mt}}{n-1} = 0,

-V^t_{mt} = V^t_{tm} (\equiv V_m).

Substituting the above formula into (2.11) to obtain

V^\alpha_{\gamma m} = \delta^\alpha_{\gamma} \frac{V_{m}}{n-1} - \delta^\beta_{\gamma} \frac{V_{n-1}}{n-1} \delta^\beta_{m} \frac{\Omega}{n-1},

where \Omega = \frac{V_{n-1}}{n-1} = \frac{V_{n-1}}{n-1}.

This proves that the transformation T_v must be T_\Omega; then, according to theorem 2, \Omega = \Omega_{jdx^j} must be a closed differential form; i.e., \Omega = \Lambda.

The analogous result for S_{ik} is stronger.

**THEOREM 5.** The transformation T_v that makes the Ricci tensor S_{ik} of arbitrary U invariant must be the transformation T_\Lambda.

**PROOF.** In fact,

S^v_{ik} = S_{ik}(U) = \bar{U}^S_{ik,s} - \bar{U}^S_{it} \bar{U}^t_{st} + \frac{1}{n-1} \bar{U}^S_{is} \bar{U}^t_{tk}

= (U^S_{ik,s} + V^S_{ik,s}) - (U^S_{it} + V^S_{it}) (U^t_{sk} + V^t_{sk}) + \frac{1}{n-1} (U^S_{is} + V^S_{is}) (U^t_{tk} + V^t_{tk})

= S_{ik} + F_{ik},

where,

F_{ik} = V^S_{ik,s} - U^S_{it} \bar{U}^t_{sk} - V^S_{it} \bar{U}^t_{sk} - V^S_{it} \bar{U}^t_{sk} + \frac{1}{n-1} U^S_{is} V^t_{tk}

+ \frac{1}{n-1} V^S_{is} U^t_{tk} + \frac{1}{n-1} V^S_{ik} V^t_{tk}.

According to the condition of the theorem, S^v_{ik} = S_{ik}, and we obtain F_{ik} = 0 (for arbitrary U). Therefore,

\[
\frac{\partial F_{ik}}{\partial U^Y_{\alpha \beta}} = - \delta^\alpha_{\gamma} \frac{\delta^\beta_{t} V^t_{sk}}{Y_{l} \gamma \gamma} + \delta^{t}_{\gamma} \delta^{\alpha}_{\gamma} V^S_{it} + \frac{1}{n-1} \delta^\alpha_{\gamma} \delta^\beta_{t} V^t_{tk} + \frac{1}{n-1} \delta^\alpha_{\gamma} \delta^\beta_{t} V^t_{tk}

+ \frac{1}{n-1} \delta^\alpha_{\gamma} \delta^\beta_{t} V^S_{is} = 0.
\]

In the above formula, let \alpha = \gamma. Adding from 1 to n we obtain

\[
-V^\beta_{ik} - \delta^\beta_{k} V^t_{it} + \frac{1}{n-1} \delta^\beta_{k} V^t_{tk} + \frac{n-1}{n-1} \delta^\beta_{k} V^S_{is} = 0,
\]

\[
-V^\beta_{ik} + \frac{1}{n-1} \delta^\beta_{k} V^t_{tk} + \frac{1}{n-1} \delta^\beta_{k} V^t_{tk} = 0,
\]

\[
V^\beta_{ik} = \delta^\beta_{k} \left( \frac{1}{n-1} V^t_{tk} \right) + \delta^\beta_{k} \left( \frac{1}{n-1} V^t_{tk} \right).
\]

(2.12)

In the above formula, let \beta = i; again, adding from 1 to n we obtain
\[ V_k = V^{t}_{tk} = \frac{n}{n-1} V^{t}_{tk} + \frac{1}{n-1} V^{t}_{kt} = \frac{n}{n-1} V_k + \frac{1}{n-1} V^{t}_{kt}, \]

\[ V_k = -V^{t}_{kt} = -V^{t}_{tk}. \]

Substituting the above formula into (2.12) we obtain (note \( \Omega_k = \frac{1}{n-1} V_k \))

\[ V_{ik} = \delta_{ik} \Omega_k - \delta_{ik} \Omega^t. \]

From theorem 2, it follows that \( \Omega = \Omega_j dx^j \) is a closed differential form; namely \( T_v \) must be \( T_\lambda \).

Moreover, we have the following

**THEOREM 6.** The transformation \( T_v \) that makes scalar curvature \( S = g^{ik} S_{ik} \) of arbitrary \( U \) invariant must satisfy the following system of equations

\[ g^{ik} (V^\beta_{\gamma k} - \frac{1}{n-1} \delta^\beta_{\gamma} V_k) + g^{ib} (V^a_{i\gamma} + \frac{1}{n-1} \delta^a_{\gamma} V_i) = 0, \quad (2.13) \]

where \( V_k = V^{t}_{tk} = -V^{t}_{kt} \).

**PROOF.** From (2.7), we have

\[ S = g^{ik} S_{ik} = g^{ik} U^{s}_{ik,s} - g^{ik} U^{s}_{it,sk} + \frac{1}{n-1} g^{ik} U^{s}_{is,tk} \]

\[ = g^{ik} (U^{s}_{ik,s} + V^{s}_{ik,s}) - g^{ik} (U^{s}_{it} + V^{s}_{it}) (U^{t}_{sk} + V^{t}_{sk}) \]

\[ + \frac{1}{n-1} g^{ik} (U^{s}_{is} + V^{s}_{is}) (U^{t}_{tk} + V^{t}_{tk}) = S + F, \]

where,

\[ F = g^{ik} V^{s}_{ik,s} - g^{ik} U^{s}_{it,sk} - g^{ik} V^{s}_{it,sk} - g^{ik} V^{s}_{it} \]

\[ + \frac{1}{n-1} g^{ik} U^{s}_{is,tk} + \frac{1}{n-1} g^{ik} V^{s}_{is,tk} + \frac{1}{n-1} g^{ik} V^{s}_{is} (2.14) \]

Multiply the above formula by \( g^{ab} \), adding from 1 to \( n \) for \( \alpha \) and \( \beta \) to obtain

\[ -g^{ab} g^{ik} V^\beta_{\gamma k} = g^{ab} g^{ib} V^a_{i\gamma} + \frac{1}{n-1} g^{ab} g^{ik} V^a_{\gamma} + \frac{1}{n-1} g^{ab} g^{ib} V^a_{i\gamma} (2.14) \]

\[ -V^k_{\gamma k} - V^a_{\gamma} + \frac{1}{n-1} V^{t}_{\gamma t} + \frac{1}{n-1} V^{s}_{\gamma s} = 0. \]
From this we obtain,
\[ V_\gamma = V_{\gamma t}^t = -V_{\gamma t}^t. \]

Substituting the above formula into (2.14), we have
\[ g^{ak} (V_{\alpha k}^\beta - \frac{1}{n-1} \delta_\gamma^\beta V_k^\alpha) + g^{i\beta} (V_{i\alpha}^\gamma + \frac{1}{n-1} \delta_\gamma^\alpha V_i^\gamma) = 0. \]

REMARK. Let \( \Omega = \Omega_j dx^j \) be a 1-differential form. It is easy to prove that
\[ V_{ik}^\beta = \delta_{i k}^T \Omega_k^\beta \]
is a second order differentiable covariant tensor field with vector value. By the following computation, we know it satisfies (2.13).

\[ V_{\alpha k}^\beta = \frac{1}{n-1} \delta_\gamma^\beta V_k^\alpha = (\delta_\gamma^\beta \Omega_k^\alpha - \delta_\gamma^\beta \Omega_k^\alpha) - \frac{1}{n-1} \delta_\gamma^\beta \Omega_k^\alpha \]
\[ = \delta_\gamma^\beta \Omega_k^\alpha - \delta_\gamma^\beta \Omega_k^\alpha - \frac{n}{n-1} \delta_\gamma^\beta \Omega_k^\alpha + \frac{1}{n-1} \delta_\gamma^\beta \Omega_k^\alpha = -\delta_\gamma^\beta \Omega_k^\alpha \]
\[ V_{i\gamma}^\alpha + \frac{1}{n-1} \delta_\gamma^\alpha V_i^\gamma = (\delta_\gamma^\alpha \Omega_i^\gamma - \delta_\gamma^\alpha \Omega_i^\gamma) + \frac{1}{n-1} \delta_\gamma^\alpha \Omega_i^\gamma = \delta_\gamma^\alpha \Omega_i^\gamma \]
\[ g^{ak} (V_{\alpha k}^\beta - \frac{1}{n-1} \delta_\gamma^\beta V_k^\alpha) + g^{i\beta} (V_{i\alpha}^\gamma + \frac{1}{n-1} \delta_\gamma^\alpha V_i^\gamma) \]
\[ = g^{ak} (-\delta_\gamma^\beta \Omega_k^\alpha) + g^{i\beta} (\delta_\gamma^\alpha \Omega_i^\gamma) = -g^{\alpha\beta} \Omega_k^\alpha + g^{\alpha\beta} \Omega_i^\gamma = 0. \]

From the remark of theorem 3, it follows that although \( T_\gamma \) satisfies (2.13), perhaps it does not make scalar curvature \( S \) invariant. Therefore, (2.13) is only a necessary condition under which the transformation \( T_\gamma \) makes scalar curvature \( S \) invariant.

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