DOT PRODUCT REARRANGEMENTS

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ABSTRACT. Let $a = (a_n)$, $x = (x_n)$ denote nonnegative sequences; $x = (x_{\pi(n)})$ denotes the rearranged sequence determined by the permutation $\pi$, $a \cdot x$ denotes the dot product $\sum_{n=1}^{\infty} a_n x_n$, and $S(a, x)$ denotes $\{a \cdot x_{\pi} : \pi \text{ is a permutation of the positive integers}\}$. We examine $S(a, x)$ as a subset of the nonnegative real line in certain special circumstances. The main result is that if $\lim_{n \to \infty} a_n = \infty$, then $S(a, x) = [a \cdot x_{\pi}]$ for every $x_n \neq 0$ if and only if $\lim_{n \to \infty} a_{n+1}/a_n$ is uniformly bounded.

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An elementary classical result of Riemann on infinite series states that a conditionally convergent series that is not absolutely convergent can be rearranged to sum to any extended real number. A slightly similar group of questions arose in connection with certain formulas in operator theory [1, p. 181]. Namely, if we let $a = (a_n)$, $x = (x_n)$ denote any two non-negative sequences and $x_{\pi}$ denote the sequence $(x_{\pi(n)})$ where $\pi$ is any permutation of the positive integers, then what can be said about the set of non-negative real numbers $S(a, x) = \{a \cdot x_{\pi} : \pi \text{ is a permutation of the positive integers}\}$. More specifically, which subsets of the non-negative real line can be realized as the form $S(a, x)$ for some such $a$ and $x$?

Various facts about $S(a, x)$ are obvious
(1) \( S(a,x) \subseteq [0,\infty] \). The values 0 and \( \infty \) may be obtained.

(2) If \( a \) and \( x \) are strictly positive sequences or are at most finitely zero, then \( S(a,x) \subseteq (0,\infty) \).

(3) Not all subsets of \([0,\infty]\) are realizable as an \( S(a,x) \) set. This follows by a cardinality argument. If \( c \) denotes the cardinality of \([0,\infty]\), then the cardinality of the class of subsets of \([0,\infty]\) is \( 2^c \), but the cardinality of the class of sequences \( a \) and \( x \) is \( c \) and thus the cardinality of the subsets \( S(a,x) \) is less than or equal to \( c \cdot c = c \).

(4) If either \( a \) or \( x \) is finitely non-zero then \( S(a,x) \) is countable.

(5) An example: if \( a = (0,2,0,2,...) \) and \( x = (3^{-n}) \), then \( S(a,x) \) is precisely the Cantor set except for those non-negative real numbers whose ternary expansion consists of a tail of 0's or a tail of 2's (i.e., a subset of the rational numbers).

It seems too ambitious to consider the general question at this time. For this reason we shall restrict our attention to the cases when \( a \) is a non-decreasing sequence and \( x \) is a non-increasing sequence.

If \( a \equiv 0 \) or \( x \equiv 0 \), the problem is trivial and \( S(a,x) = \{0\} \). If \( a \neq 0 \) and \( x \neq 0 \), the problem is trivial and \( S(a,x) = \{\infty\} \). If \( a \) is bounded by \( M \), then \( S(a,x) \subseteq [0, M \sum x_n] \). In any case, hereafter we shall assume \( a \nrightarrow \infty \) and \( x_n \neq 0 \), unless otherwise specified.

The Lemma that follows is a well-known fact, but we give a proof for completeness and because the proof contains some of the ideas used in the main result.

**Lemma.** If \( a \nrightarrow \infty \) and \( x \nrightarrow 0 \), then \( S(a,x) \subseteq [a \cdot x, \infty) \). In addition, \( a \cdot x \in S(a,x) \), and if \( a \nrightarrow \infty \) and \( x_n \neq 0 \) for all \( n \) or if \( a \nrightarrow \infty \) and \( a_n > 0 \) for some \( n \) and \( x_n \neq 0 \), then \( \infty \in S(a,x) \).

**Proof.** It suffices to show that for every permutation \( \pi \) of the positive integers, we have \( a \cdot x \leq \sum a_n x_{\pi(n)} \) or, equivalently, \( a \cdot x \leq \sum a_{\pi(n)} x_n \) for every \( \pi \). The rest of the lemma is clear.

Define \( \pi \) in terms of \( \pi \) as follows. Set...
It is straightforward to verify that \( \pi_1 \) is also a permutation of the positive integers (one-to-one and onto) which fixes 1. We assert that \( a_\pi \cdot x \leq a_\pi \cdot x \) for all \( \pi \).

To see this, note that \( \pi(1) \geq 1 \) and \( \pi^{-1}(1) \geq 1 \). Hence \( a_{\pi(1)} - a_1 \geq 0 \) and \( x_1 - x_{\pi^{-1}(1)} \geq 0 \). Therefore

\[
\sum_{n=1}^{k} (a_{\pi(n)} - a_{\pi_1(n)}) x_n = \sum_{n=1}^{k} a_{\pi(n)} x_n - a_{\pi_1(n)} x_1 \geq 0.
\]

Proceeding inductively, we obtain a sequence of permutations \( \pi_k \) that fix 1, 2, \ldots, \( k \) for which \( a_{\pi_k} \cdot x \leq a_{\pi_k} \cdot x \). Hence, for every \( k \),

\[
\sum_{n=1}^{k} a_{\pi_k(n)} x_n = \sum_{n=1}^{k} a_{\pi_k(n)} x_n \leq a_{\pi_k} \cdot x \leq a_{\pi_k} \cdot x.
\]

Letting \( k \to \infty \), we obtain \( a\cdot x \leq a_{\pi} \cdot x \).

The main question of this paper is: for which \( a, x \) with \( a_n \to \infty \) and \( x_n \to 0 \) is \( S(a, x) = [a \cdot x, \infty) \)?

The main result of this paper gives a partial answer. Namely, we can characterize which \( a_n \to \infty \) have the property that \( S(a, x) = [a \cdot x, \infty) \) for every \( x \) such that \( x_n \neq 0 \).

On first sight, it might appear that \( S(a, x) \) can never be \( [a \cdot x, \infty) \) or that it is quite rare. The first result in this direction was that if \( a_n = n \) for every \( n \), then \( S(a, x) = [a \cdot x, \infty) \) for every \( x \) such that \( x_n \neq 0 \). That \( S(a, x) \) may not be \( [a \cdot x, \infty) \) was first decided by an example due to Robert Young. Namely, let \( a_n = 2^n \) and \( x_n = 2^{-n+1} \). Both results are unpublished. The succeeding results and techniques are due to the work of the authors in collaboration with Hugh Montgomery.
THEOREM 1. (The Main Theorem) Let \( a = (a_n) \) where \( a_n > 0 \) for every \( n \) and \( a_n \to \infty \). Consider the following conditions:

1. \( a_{n+1}/a_n \) is bounded.
2. For the non-negative sequence \( x = (x_n) \), there exist subsequences \( (a_m) \) and \( (x_m) \) of \( a \) and \( x \) respectively such that
   a. \( a_n x_n \to 0 \) as \( k \to \infty \), and
   b. \( \sum_{k} a_n x_n = \infty \).
3. \( S(a,x) = [a \cdot x, \infty) \).

Then (1) implies (2) for every strictly positive sequence \( x = (x_n) \) that tends to 0. Also if \( a_n \to \infty \) and \( x_n \to 0 \) where \( a_n x_n \neq 0 \) for all \( n \), then (2) implies (3).

PROOF. To prove that (1) implies that (2) holds for every strictly positive sequence \( x = (x_n) \) that tends to 0, suppose \( a_{n+1}/a_n \leq M \) for all \( n \). We assert that for every positive integer \( k \), there exist arbitrarily large positive integers \( n_k \) and \( m_k \) for which \( (k+1)^{-1} \leq a_n x_n \leq M k^{-1} \). If this assertion were true, then clearly we could choose two strictly increasing subsequences of positive integers \( (n_k) \) and \( (m_k) \) such that \( a_n x_n \to 0 \) as \( k \to \infty \) to prove the assertion.

For each fixed positive integer \( k \), \( (k+1)^{-1} \leq a_n x_n \leq M k^{-1} \) if and only if \( x_m \notin [(a_n (k+1))^{-1}, M(a_k)^{-1}] \). All we need show is that there exist arbitrarily large \( n,m \) for which \( x_m \notin [(a_n (k+1))^{-1}, M(a_k)^{-1}] \).

Suppose to the contrary that there exists a positive integer \( N \) for which \( x_m \notin [(a_n (k+1))^{-1}, M(a_k)^{-1}] \) for every \( n,m \geq N \). In other words, for every \( m \geq N \), \( \bigcup_{n \geq N} [(a_n (k+1))^{-1}, M(a_k)^{-1}] \). (Note: This would imply that \( \bigcup_{n \geq N} [(a_n (k+1))^{-1}, M(a_k)^{-1}] \) cannot contain any interval of the form \((0,\epsilon)\) for some \( \epsilon > 0 \), since \( x_m \to 0 \) as \( m \to \infty \). However, this is not the case. Indeed, the proof below can be used to show that for every \( N \), there exists \( \epsilon > 0 \) such that \( (0,\epsilon) \subseteq \bigcup_{n \geq N} [(a_n (k+1))^{-1}, M(a_k)^{-1}] \).)

For each \( n \geq N \), let \( n_m \) denote the least positive integer \( n \) such that \( M(a_n+1)^{-1} < x_m \), which exists since \( a_n \to \infty \) as \( n \to \infty \) and hence \( M(a_n+1)^{-1} \to 0 \).
as $n \to \infty$. For $m$ sufficiently large, we have $M(a_{n+1}^k)^{-1} \leq x_m \leq M(a_n^k)^{-1}$.

Also, since $M(a_{n+1}^{k+1})^{-1} < x_m$ and $x_m \to 0$ as $m \to \infty$, we have $m \to \infty$ implies $n_{m+1} \to \infty$ and hence $n_m \to \infty$. Therefore $n_m \geq N$ for all $m$ sufficiently large, and for these $m$, $x_m \not\in [(a_{n_m})/(k+1), M(a_n^k)]$. Hence, for infinitely many $m$, we have $x_m \leq M(a_n^k)$ and $x_m \not\in [(a_{n_m})/(k+1), M(a_n^k)]$. Therefore, for infinitely many $m$, we have $M(a_{n+1}^k)^{-1} < x_m < (a_n^k)^{-1}$. This implies that $M(a_{n+1}^{k+1})^{-1} < (a_n^{k+1})^{-1}$ for infinitely many $m$, or equivalently, $a_{n+1}/a_n > M(k+1)/k > M$ for infinitely many $m$, which contradicts our assumption that $a_{n+1}/a_n < M$ for all $n$. Hence (2) is proved.

To prove (2) \rightarrow (3) whenever $a_\infty$ and $x_\infty$, suppose (2) holds for $a$ and $x$, so that there exist subsequences $(a_n^1)$ and $(x_1^1)$ such $a_n^1 \to 0$ as $n \to \infty$, and $\sum_{k=1}^{n} a_k x_k = \infty$. We first assert that without loss of generality we may assume that $a_n^1 x_1^1 \leq \infty$. To see this suppose $a_n^1 x_1^1 = \infty$. Then by the lemma we have that $S(a^1, x^1) \in \{\infty\}$, and hence (3) holds.

Assuming that $\sum_{n=1}^{\infty} a_n x_n < \infty$, we next assert that without loss of generality we can assume that $n_k > m_k$ for every $k$. To see this, let $Z_1$ denote the set \{$n_k > m_k$\} and let $Z_2$ denote the set \{$m_k < n_k$\}. Then

$$\sum_{n=1}^{\infty} a_n x_n = \sum_{k \in Z_1} a_n^k m_k + \sum_{k \in Z_2} a_n^k m_k$$

But $\sum_{k \in Z_2} a_n^k m_k < \sum_{k \in Z_2} a_n^k m_k < \sum_{k \in Z_1} a_n^k m_k$. Therefore $\sum_{n=1}^{\infty} a_n x_n < \infty$. Let $Z_1$ determine subsequences of $(n_k)$ and $(m_k)$, which for simplicity we again call $(n_k)$ and $(m_k)$, respectively, by taking only those entries $n_k, m_k$ (in increasing order) for which $k \in Z_1$. This gives us subsequences $(a_n^k)$ and $(x_1^k)$ of $a$ and $x$ which satisfy conditions $a$ and $b$ in the 2nd condition of the theorem, and in addition satisfy $n_k > m_k$ for all $k$.

Next we assert that without loss of generality we may assume $n_k \neq m_j$ for all $k,j$. To see this, note that we have $n_k > m_k$ for all $k$ and that $<n_k>$ and $<m_k>$ are strictly increasing (a property of subsequences). Therefore if $n_k = m_j$ for
some \( k, j \), then \( k < j \) and \( n_k \neq m_j \) for all \( i \neq j \). That is, \( n_k \) can occur at most once among the \( m_j \)'s. Put \((n_1, m_1), \ldots, (n_{k_1}, m_{k_1}) \in S_1 \) where \( k_1 + 1 \) is the least positive integer such that \( m_{k_1 + 1} = n_k \) for some \( k < k_1 + 1 \). Put \((n_{k_1 + 1}, m_{k_1 + 1}), \ldots, (n_{k_2}, m_{k_2}) \in S_2 \) where \( k_2 + 1 \) is the least positive integer, if it exists, such that \( m_{k_2 + 1} = n_k \) for some \( k_1 + 1 \leq k < k_2 + 1 \). Put \((n_{k_2 + 1}, m_{k_2 + 1}), \ldots, (n_{k_3}, m_{k_3}) \in S_1 \) such that \( k_3 + 1 \) is the least positive integer, if it exists, such that \( m_{k_3 + 1} = n_k \) for some \( k < k_2 + 1 \) or \( k_2 < k < k_3 + 1 \). Continuing in this way, if no such least positive integer exists, then either \( S_1 \) or \( S_2 \) is finite. Otherwise both \( S_1, S_2 \) are infinite. For either case, no \( n_k = m_j \) when both \((n_k, m_k), (n_j, m_j) \) \( \in S_1 \) or \( S_2 \). Then clearly \( S_1, S_2 \) is a disjoint partition of the set of all \((n_k, m_k)\) and in each set, no \( n_k \) appears as an \( m_j \). Therefore \( \sum n_k \frac{a_x}{m_k} \) and \( \sum m_k \frac{a_x}{n_k} \), and so either \( \sum n_k \frac{a_x}{m_k} = \infty \) or \( \sum m_k \frac{a_x}{n_k} = \infty \). Choosing \( S_1 \) or \( S_2 \) accordingly we produce the sequence \((n_k, m_k)\) with the desired properties, (i.e., satisfying a) and b) in Theorem 1 and also satisfying \( n_k \neq m_j \) for all \( k, j \) and \( n_k > m_k \) for every \( k \).

Now consider the series \( \sum_k (a_n - a_m)(x_n - x_m) \). Since \( n_k > m_k \), we have

\[
0 \leq a_n - a_m \leq a_n \quad \text{and} \quad 0 \leq x_n - x_m \leq x_n,
\]

and so \( 0 \leq (a_n - a_m)(x_n - x_m) \leq (a_n - a_m)x_n \). Furthermore, since \( \sum_k a_n x_n = \infty \), \( a_n x_n \geq 0 \), \( \sum_k a_n x_n \leq a \cdot x \leq \infty \), \( \sum_k a_n x_n \leq a \cdot x \leq \infty \), and \( \sum_k a_n x_n \leq a \cdot x \leq \infty \), we have

\[
\sum_k (a_n - a_m)(x_n - x_m) = \sum_k (a_n x_n + a_n x_n - a_n x_n - a_n x_n) = \infty.
\]

We shall now show that for every \( \varepsilon > 0 \), there exists a subsequence \( (k) \) of positive integers such that \( \varepsilon = \sum_k (a_n - a_m)(x_n - x_m) \). This follows from the following more general fact.

Suppose \((d(k))\) is a non-negative sequence for which \( d(k) \to 0 \) as \( k \to \infty \) and \( \sum d(k) = \infty \). We assert that very every \( \varepsilon > 0 \), there exists a subsequence \( (k) \) such that \( \varepsilon = \sum_k d(k) \). The proof of this fact proceeds along the same lines as the proof of Riemann's theorem on rearrangements of conditionally convergent series. Fix
\( \varepsilon > 0 \) and choose \( n_1 > N_1 \) so that \( d(k) < \varepsilon \) for every \( k > N_1 \) and so that \( n_1 \) is the greatest integer greater than \( N_1 \) such that \( \sum_{k=N_1}^{n_1} d(k) < \varepsilon \). Hence \( \sum_{k=1}^{n_1} d(k) < \varepsilon \leq \sum_{k=1}^{n_1+1} d(k) \). This can be done since \( d(k) \to 0 \) as \( k \to \infty \) and \( d(k) = \infty \).

Choose \( N_2 > n_1 \) so that \( d(k) < (\varepsilon - \sum_{k=1}^{N_2} d(k))/2 \) for every \( k > N_2 \) and then choose \( n_2 \) to be the largest integer greater than \( N_2 \) such that \( d(k) < \varepsilon - \sum_{k=1}^{n_2} d(k) \). Hence \( \sum_{k=1}^{n_1} d(k) < \varepsilon - \sum_{k=1}^{n_1+1} d(k) \). Proceeding inductively in this way, we obtain sequences \( (N_p) \) and \( (n_p) \) of positive integers for which

\[
\sum_{k=1}^{n_p} d(k) < \varepsilon - \sum_{k=1}^{n_{p+1}} d(k) \frac{p-1}{2^{p-1}} \quad \text{for every } p \text{ and every } k > N_p, \quad \text{and}
\]

\[
\sum_{k=1}^{n_p} d(k) < \varepsilon - \sum_{k=1}^{n_{p+1}} d(k) \frac{p-1}{2^{p-1}} \quad \text{for every } p \text{ and every } k > N_p.
\]

This implies that

\[
0 < \varepsilon - \sum_{q=1}^{n_p} d(k) \leq d(n_p + 1) \leq (\varepsilon - \sum_{q=1}^{n_{p+1}} d(k))/2^{p-1} \leq \varepsilon/2^{p-1} \to 0 \text{ as } p \to \infty.
\]

Therefore \( \varepsilon = \sum_{q=1}^{n_p} d(k) \). Hence, if we choose \( (k_n) \) to be the strictly increasing sequence of positive integers \( k \), where \( k \) is taken to range over the set \( \bigcup_{p=1}^{\infty} \{ k : N_p < k < n_p \} \), we have \( \varepsilon = \sum_{k=n_k}^{d(k)} \).

Applying this result to the sequence \( (a_k - a_k)(x_k - x_k) \), since it is non-negative, tends to 0, and sums to \( \infty \), we obtain that for every \( \varepsilon > 0 \), there exist subsequences of \( (n_k) \) and \( (m_k) \), which we shall again denote by \( (n_k) \) and \( (m_k) \), for which \( \varepsilon = \sum_{k=n_k}^{d(k)} \).

Now recall that we wish to show that \( S(a,x) = [a\cdot x, \infty] \). We already know \( a\cdot x \) and \( \in\in [a\cdot x, \infty] \). Suppose \( a\cdot x < r < \infty \). It suffices to show \( r \in S(a,x) \).

Let \( \varepsilon = r - a\cdot x \) and choose subsequences which we again call \( (n_k) \) and \( (m_k) \) so that
\[ \epsilon = \sum \frac{a_n - a_m}{m_k} (x_n - x_m). \]

We now choose \( \pi \), the requisite permutation on \( \mathbb{Z}^+ \), as follows. Let \( \pi(n_k) = m_k \) and \( \pi(m_k) = n_k \) for each \( k \), and let \( \pi \) fix all other integers \( n \) (i.e., those \( n \) for which \( n \neq n_k, m_k \) for every \( k \)). The permutation \( \pi \) is well-defined since \( n \neq m_i \) for every \( i, j \). Let \( Z_n \) denote the set \{ \( n \) : \( n = n_k \) or \( n = m_k \) for some \( k \) \}. Hence \( \pi(n) = n \) for all \( n \not\in Z_n \). Then

\[
\sum_{n \not\in Z_n} a_n x_{\pi(n)} = \sum_{n \not\in Z_n} a_n x_n + \sum_{n \not\in Z_n} (a_n x_m + a_m x_n) + (a_n - a_m)(x_n - x_m).
\]

\[
= \sum_{n \not\in Z_n} a_n x_n + \sum_{n \not\in Z_n} (a_n - a_m)(x_n - x_m).
\]

\[
= a \cdot x \in \epsilon = r,
\]

and so \( \epsilon \in S(a, x) \), which proves (3).

Q.E.D.

**Theorem 2.** Let \( a = a_n \) where \( a_1 > 0 \) and \( a_n \uparrow \infty \). Then \( \frac{a_{n+1}}{a_n} \) is bounded if and only if, for every \( x = x_n \) for which \( x_n \neq 0 \), \( S(a, x) = [a \cdot x, \infty) \).

**Proof.** If \( \frac{a_{n+1}}{a_n} \) is bounded, then by Theorem 1, if \( x_n \neq 0 \), then \( x = x_n \) satisfies condition (2) of the theorem. Also by Theorem 1, since \( a_n \uparrow \infty \) and \( a_1 > 0 \), condition (3) of the theorem is satisfied by \( x \). That is, \( S(a, x) = [a \cdot x, \infty) \).

Conversely, if \( S(a, x) = [a \cdot x, \infty) \) for every \( x = x_n \) for which \( x_n \neq 0 \), we claim that \( a_{n+1}/a_n \) must remain bounded.

Suppose to the contrary that \( a_{n+1}/a_n \) is not bounded. Let \( h(n) \) denote the least positive integer \( k \) for which \( k \geq n \) and \( a_{k+1}/a_k \geq 4^n \). Clearly \( h(n) \) is a non-decreasing function of \( n \). Define \( x_n = (a_{h(n)3^n})^{-1} \). Then \( x_n \neq 0 \). Letting \( x = x_n \), we claim that \( S(a, x) \neq [a \cdot x, \infty) \). In fact, we claim that \( a \cdot x < 1 \) but \( l \not\in S(a, x) \). Indeed, \( a \cdot x = \sum a_n x_n = \frac{\sum a_n (a_{h(n)3^n})^{-1}}{3^{-n}} \leq \frac{3^{-n}}{3^{-n}} = 1/2 < 1 .\) Furthermore, letting \( \pi \) be any permutation of \( \mathbb{Z}^+ \), if \( \pi^{-1}(k) > h(k) \) for some \( k \), then

\[
\sum a_n x_{\pi(n)} \geq a_{\pi^{-1}(k)} x_k \geq a_{h(k) + 1} x_k = a_{h(k) + 1} (a_{h(k)3^k})^{-1} \geq 4^k 3^{-k} > 1.
\]
On the other hand, if $\pi^{-1}(k) \leq h(k)$ for every $k$, then

$$\sum_{n} a_{\pi(n)} x_{n} = \sum_{n} a_{\pi^{-1}(k)} x_{n} \leq a_{h(n)} x_{n} = \int_{0}^{\infty} 3^{-n} = 1/2 < 1.$$  

In any case, $\sum_{n} a_{\pi(n)} x_{n} \neq 1$, hence $1 \notin S(a, x)$. Q.E.D.

NOTE. In the proof of Theorem 1, each time we constructed a permutation $\pi$ to solve the equation $\sum_{n} a_{\pi(n)} = r$, it sufficed to use only disjoint 2-cycles. That is, each such $\pi$ that we constructed was the product of disjoint 2-cycles. This seems odd and leads us to ask if there are any circumstances in which the use of infinite-cycles or n-cycles yields more. In other words, is it always true that $S(a, x)$ is the same as $\{ \sum_{n} a_{\pi(n)} : \pi$ is a permutation of $\mathbb{Z}^+$ which is a product of disjoint 2-cycles $\}$?

The following question seems likely to have an affirmative answer. If so, this would give a characterization for those sequences $a$ and $x$ where $a_n \rightarrow \infty$, $a_1 > 0$, and $x_n \neq 0$, which satisfy $S(a, x) = \{ a \cdot x, x \}$. However, it remains unsolved.

QUESTION 1. If $a$ and $x$ are as above, does (3) $\Rightarrow$ (2) in Theorem 1?

Finally, we wish to point out that Theorems 1 and 2 imply analogous theorems in which $a$ and $x$ switch roles. Indeed, the proofs of the following two corollaries follow naturally along the same lines as those of Theorems 1 and 2.

COROLLARY 3. Let $x = (x_n)$ where $x_n > 0$ for all $n$, and $x_n \rightarrow 0$ as $n \rightarrow \infty$.

Consider the following conditions.

1. $x_n/x_{n+1}$ is bounded below.
2. For the non-negative sequence $a = (a_n)$, there exist subsequences $(\pi_n)$ and $(\pi_x)$ of $a$ and $x$, respectively, such that
   a) $a_{\pi_n} x_{\pi_n} \rightarrow 0$ as $k \rightarrow \infty$, and
   b) $\sum_{k} a_{\pi_n} x_{\pi_n} = \infty$.

Then (1) implies that (2) holds for every strictly positive sequence $a = (a_n)$ that tends to $\infty$. 
COROLLARY 4. Let \( x = (x_n) \) be a non-negative sequence. Then \( x_n/x_{n+1} \) is bounded below if and only if, for every \( a = (a_n) \) for which \( a_n \uparrow \) and \( a_1 > 0 \),
\[
S(a,x) = \{ a \cdot x, \infty \}.
\]

QUESTION 2. Is there anything to be said about the qualitative nature of \( S(a,x) \)? Is it always a Borel set, measurable, \( F_\sigma, G_\sigma \)?

REFERENCE