A NOTE ON AN EXTENSION OF LINDELÖF'S THEOREM TO MEROMORPHIC FUNCTIONS

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ABSTRACT. S. M. Shah [3] has given an extension of Lindelöf's Theorem to meromorphic functions. He also obtained an expression for the characteristic function of a meromorphic function of integer order. In this note we give estimates for $\log |f(re^{i\theta})|$ of such functions.

KEY WORDS AND PHRASES. meromorphic functions, proximate order, slowly changing functions.

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1. INTRODUCTION.

In [3; theorem 1] S. M. Shah obtained an expression for the characteristic function $T(r,f)$ of a meromorphic function $f(z)$ of integer order $\rho$. Following the argument of Cartwright [2; theorem 45, 46] we can obtain the following results for $\log |f(re^{i\theta})|$. We write

$$n(r) = n(r,1/f) + n(r,f); \quad N(r) = N(r,1/f) + N(r,f).$$

Since $\rho$ is a positive integer, we can write $f(z)$ in the form (see [3])

$$f(z) = z^k \exp(c z^\rho + \ldots) \prod_{n=1}^{\infty} E(z/a_n, \rho) \prod_{n=1}^{\infty} E(z/b_n, \rho). \quad (1.1)$$

Let $\rho(r)$ be a proximate order [3] for $N(r)$ and let $n_L = \limsup \frac{n(r)}{r^\rho L(r)}$,

where $L(r)$ is a slowly changing function.

2. MAIN RESULTS.

THEOREM. Let $f(z)$ be a meromorphic function of integer order $\rho > 0$ and let

$$S(r) = c + \frac{1}{\rho} \sum_{n} a_n^{-\rho} - \frac{1}{\rho} b_n^{-\rho},$$

where $a_n$ and $b_n$ are the zeros of $f(z)$.
i. Suppose $n_L < \infty$. Then for every $\eta > 0$, there is a $K(\rho, \eta)$ such that for every $\epsilon > 0$,

$$\log |f(re^{i\theta})| - \text{Re}(r^\rho e^{i\theta} S(r)) < K(\rho, \eta) (n_L + \epsilon) r^\rho L(r)$$

(2.1)

for $0 \leq r \leq R$, except perhaps in circles the sum of whose radii is less than $\eta R$, provided that $R > R_0(\epsilon, \eta)$. 

ii. Suppose $N(r)$ is of order $\rho$. Then there is a $K(\rho, \eta)$ such that

$$\log |f(re^{i\theta})| - \text{Re}(r^\rho e^{i\theta} S(r)) < K(\rho, \eta) r^\rho$$

(2.2)

for $0 < r < R$, except perhaps in circles the sum of whose radii is less than $R$, provided that $R > R_0(\eta)$. 

iii. Let $\lim \sup \log N(r)/\log r = c_1 < \rho$ and let $c_1 < c_2 < \rho \leq 1 + c_2$. Then for every $\eta > 0$, there is a $K(c_2, \eta)$ such that

$$\log |f(re^{i\theta})| - \text{Re}(r^\rho e^{i\theta} S(r)) < K(c_2, \eta) r^{c_2}$$

for $0 \leq r \leq R$, except perhaps in circles the sum of whose radii is less than $\eta R$, provided that $R > R_0(c_2, \eta)$. The proof depends on the following lemma of Cartan (see [1, p.46], also [2, pp.73-77]):

**Lemma (H. Cartan).** Let $p(z) = \prod_{k=1}^{n} (z-z_k)$; for any positive $H$, the inequality

$$|p(z)| > (H/e)^n$$

holds outside at most $n$ circles the sum of whose radii is at most $2H$.

We omit the details of the proof of the theorem.

**REFERENCES**