INFINITE MATRICES AND ABSOLUTE ALMOST CONVERGENCE

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ABSTRACT. In 1973, Stieglitz [5] introduced a notion of $F_B$-convergence which provided a wide generalization of the classical idea of almost convergence due to Lorentz [1]. The concept of strong almost convergence was introduced by Maddox [3] who later on generalized this concept analogous to Stieglitz's extension of almost convergence [4]. In the present paper we define absolute $F_B$-convergence which naturally emerges from the concept of $F_B$-convergence.

KEY WORDS AND PHRASES. Infinite matrices, almost convergence, strong almost convergence, $F_B$-convergence, absolute $F_B$-convergence.

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1. INTRODUCTION.

Let $\ell_\infty$, $c$, and $c_0$ denote respectively the Banach spaces of bounded, convergent, and null sequences $x = (x_k)$ of complex numbers with norm $||x|| = \sup_k |x_k|$, and let $v$ be the space of sequences of bounded variation, that is,

$$v = \{x: ||x|| = \sum_{k=0}^m |x_k - x_{k-1}| < +\infty, x_{-1} = 0\}.$$

Suppose that $B = (B_i)$ is a sequence of infinite complex matrices with $B_i = (b_{np}(i))$. Then $x \in \ell_\infty$ is said to be $F_B$-convergent [5], to the value $\lim Bx$, if

$$\lim_{n \to \infty} (B_i x)_n = \lim_{n \to \infty} \sum_{p=0}^{\infty} b_{np}(i) x_p = \lim Bx,$$

uniformly for $i = 0,1,2,\ldots$.

The space $F_B$ of $F_B$-convergent sequences depends on the fixed chosen sequence $B = (B_i)$. In case $B = B_0 = (I)$ (unit matrix), the space $F_B$ is same as $c$ and, for
\( B = B_1 = (B^{(1)}_1), \) it is same as the space \( F \) of almost convergent sequences \([1], \) where \( B^{(1)}_1 = (b^{(1)}_{np}, p) \) with

\[
b^{(1)}_{np}(i) = \begin{cases} \frac{1}{n+1}, & i \leq p \leq i + n \\ 0 & \text{otherwise} \end{cases}
\]

Maddox [4] generalized strong almost convergence by saying that \( x_p \to s[F_B] \) if and only if

\[
\sum_p b_{np}(i) |x_p - s| \to 0 \quad (n \to \infty, \text{uniformly in } i)
\]
assuming that the series in (1.1) converges for each \( n \) and \( i \).

In particular, if \( B = B_0 \), the \([F_B] \subset c; \) if \( B = B_1 \), then \([F_B] = [f], \) the space of strongly almost convergent sequences \([3], \) We shall write \( e_k = (0,0,\ldots,0,1) \) (kth entry), \( 0,\ldots, \) and \( e = (1,1,1,\ldots). \)

Let \( s \) be the space of all complex sequences and

\[
d_B = \{x \in s: \lim Bx = \lim (B_1x)_n \text{ exists for each } i\}
\]
\[
F_B = \{x \in (d_B \cap \ell_\infty): \lim t_{n}(i,x) \text{ exists uniformly in } i,
\text{ and the limit is independent of } i\},
\]

where

\[
t_n(i,x) = \begin{cases} \sum_p b_{np}(i)x_p, & (n \geq 1) \\ \sum_p \beta_{0p}(i)x_p, & (n = 0) \\ 0 & (n = -1) \end{cases}
\]

and

\[
\beta_{0p}(i) = \begin{cases} 1 & \text{if } p = i, \\ 0 & \text{otherwise}. \end{cases}
\]

Let

\[
\phi_n(i,x) = t_n(i,x) - t_{n-1}(i,x).
\]

Therefore, we have

\[
\phi_n(i,x) = \begin{cases} \sum_{p=1} b_{np}(i) x_p - b_{n-1,p}(i) x_p, & (n \geq 1) \\ \sum_{p=0} \beta_{0p}(i) x_p, & (n = 0) \end{cases}
\]

(1.2)
DEFINITION. Let $B = (B_i)$ be a sequence of infinite matrices with $B_i = (b_{np}(i))$. A sequence $x \in l_\infty$ is said to be **absolutely $F_B$-convergent** if $\sum_{n=0}^{\infty} |\theta_n(i,x)|$ converges uniformly for $i \geq 0$, and $\lim_{n \to \infty} t_n(i,x)$ which must exist should take the same value for all $i$. We denote the space of absolute $F_B$-convergent sequences by $v(B)$.

2. THE MAIN RESULT.

In this note, we denote by $(v,v(B))$ the set of matrices which give new classes of absolute $B$-conservative matrices and absolute almost $B$-conservative matrices.

Let $A$ be any infinite complex matrix for which the $p$th row-sum converges for a given $x$ for all $x$ in some class.

We have

$$A_p x = (Ax)_p = \sum_{k=0}^{\infty} a_{pk} x_k$$

and

$$B_i x_n = \sum_{p=0}^{\infty} b_{np}(i) x_p.$$  

Therefore,

$$B_i Ax_n = \sum_{p=0}^{\infty} b_{np}(i) A_p x = \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk} x_k,$$

and, assuming the interchange of order of summation can be justified (see lemma), we get that

$$B_i Ax_n = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} b_{np}(i) a_{pk} x_k$$  

(2.1)

Now, by (1.2) and (2.1), we have

$$\theta_n(i,Ax) = t_n(i,Ax) - t_{n-1}(i,Ax)$$

$$= \begin{cases} 
\sum_{p=0}^{\infty} [b_{np}(i) - b_{n-1,p}(i)] A_p x, & (n \geq 1), \\
\sum_{p=0}^{\infty} \beta_{0p}(i) A_p x, & (n = 0), \\
\sum_{k=0}^{\infty} g_{nk}(i) x_k, & (n < 0)
\end{cases}$$  

(2.2)

where
THEOREM. Let \( B = (B_i) \) be a sequence of infinite matrices with
\[
\sup_n \sum_{p=0}^\infty |b_{np}(i)| < \infty, \quad \text{for each } i.
\]

Let \( A \) be an infinite matrix. Then \( A : \nu \to \nu(B) \) if and only if

(i) \( \sup_{p,k} |\sum_{i=k}^\infty a_{pk}| < \infty, \)

(ii) there is an \( N \) such that for \( r,i = 0,1,2,... \)
\[
\sum_{n=N}^\infty |\sum_{k=0}^r g_{nk}(i)| \leq K \quad \text{(constant)},
\]

(iii) \( (a_{pk}) \in \nu(B) \) for each \( k \), and
\[
\sum_{p=0}^\infty a_{pk} \in \nu(B).
\]

Let \( A \in (\nu, \nu(B)) \). For each \( k \), let \( \sum_{k=0}^\infty a_{pk} \) be \( B \)-convergent with limit \( \alpha_k \). And let
\[
\sum_{k=0}^\infty a_{pk} \in \nu(B) \quad \text{convergent with limit } \alpha. \quad \text{(In each case, limit is taken for } p \geq 0).\]

If \( x = (x_k) \in \nu \), then
\[
\lim_{n \to \infty} (i,Ax) = \alpha \lim_{k \to \infty} x_k + \sum_{k=0}^\infty (x_k - \lim_{k \to \infty} x_k)^\alpha_k.
\]

We use the following lemma in the proof.

**LEMMA.** If either the necessity part or the sufficiency part of the theorem holds, then, for \( x \in \nu \),
\[
\sum_{p=0}^\infty b_{np}(i) \sum_{k=0}^\infty a_{pk}x_k = \sum_{k=0}^\infty x_k \sum_{p=0}^\infty b_{np}(i) a_{pk}.
\]

**PROOF.** If either \( A : \nu \to \nu(B) \) or the conditions (i)-(iv) of the theorem hold, then by partial summation, for \( x \in \nu \),
\[
\sum_{k=0}^\infty a_{pk}x_k = \sum_{k=0}^\infty d_{pk} (x_k - x_{k-1})
\]
where \( d_{pk} = \sum_{i=k}^\infty a_{ik} \). Since condition (i) holds, \( d_{pk} \) is bounded for all \( p,k \). Thus
\[
\sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk} x_k = \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} d_{pk} (x_k - x_{k-1}) \\
= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{p=0}^{\infty} b_{np}(i) d_{pk},
\]

(where the inversion is justified by absolute convergence)

\[
= \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}(i) a_{pk}
\]
since

\[
\lim_{k \to \infty} x_k \sum_{p=0}^{\infty} b_{np}(i) d_{pk} = 0.
\]

PROOF OF THEOREM. Necessity. Condition (i) follows from the fact that \(A: v \to \ell_\infty\). Since \(e_k, e \in v\), necessity of (iii) and (iv) is obvious.

It is clear that, for fixed \(p\) and \(j\),

\[
x + \sum_{k=0}^{j} a_{pk} x_k
\]
is a continuous linear functional on \(v\). We are given that, for all \(x \in v\), it tends to a limit as \(j \to \infty\) (for fixed \(p\)) and hence, by the Banach-Steinhaus Theorem [2], this limit \(A_p x\) is also a continuous linear functional on \(v\).

We observe that, although \(\sum_{n=0}^{\infty} n(i,Ax)\) is uniformly convergent in \(i\), it needs not be uniformly bounded in \(i\). For example, if \(\sum_{n=0}^{\infty} n(i,Ax) = 1\) and \(\sum_{n=0}^{\infty} n(i,Ax) = 0\) (\(n \geq 1\) and \(i\)), then \(\sum_{n=0}^{\infty} n(i,Ax)\) is uniformly convergent in \(i \geq 0\) but not uniformly bounded. Now, we can say that uniform convergence bears only on the behaviour of \(n(i,Ax)\) for sufficiently large \(n\). Thus, by definition, there is an \(m\) such that

\[
q_{m,i}(x) = \sum_{n=m}^{\infty} n(i,Ax).
\]

For \(m \geq 0\), \(i \geq 0\), \(q_{m,i}\) is a continuous seminorm on \(v\), and there is an integer \(N\) such that \(\{q_{N,i}\}_{i \geq 0}\) is pointwise bounded on \(v\). Such an \(N\) exists. For suppose not. Then for \(r = 0,1,2,\ldots\) there exists \(x_r \in v\) with

\[
\sup_{i \geq 0} q_{r,i}(x_r) = \infty.
\]

By the principle of condensation of singularities [6],

\[
\{x \in v: \sup_{i \geq 0} q_{r,i}(x) = \infty \text{ for } r = 0,1,2,\ldots\}
\]
is of second category in \( v \) and hence nonempty, i.e., there is \( x \in v \) with

\[
\sup_{i \geq 0} q_{r, i}(x) = \infty \quad \text{for} \quad r = 0, 1, 2, \ldots
\]

But this contradicts the fact that to each \( x \in v \) there exists an integer \( N_x \) with

\[
\sup_{i \geq 0} q_{N_x, i}(x) < \infty.
\]

Now, by another application of the Banach-Steinhaus Theorem, there exists a constant \( M \) such that

\[
q_{N_x, i}(x) \leq M |x|.
\]

Apply (2.3) with \( x = (x_k) \) defined by \( x_k = 1 \) for \( k \leq r \) and \( 0 \) for \( k > r \). Hence (ii) must hold.

Sufficiency. Suppose that the conditions (i)-(iv) hold and that \( x \in v \). We have defined \( v(B) \) as a subspace of \( l_\infty \). Thus, in order to show that \( Ax \in v(B) \), it is necessary to prove that \( Ax \) is bounded. By virtue of condition (i), this follows immediately.

Now, it follows from (iv) and the lemma that

\[
\sum_{k=0}^{\infty} g_{nk}(i)
\]

converges for all \( i, n \). Hence, if we write

\[
h_{nk}(i) = \sum_{k=0}^{\infty} g_{nk}(i),
\]

then \( h_{nk}(i) \) is defined, also for fixed \( i, n \),

\[
h_{nk}(i) \to 0 \quad \text{as} \quad k \to \infty.
\]

Now condition (iv) gives us that

\[
\sum_{n=0}^{\infty} |h_{n0}(i)|
\]

converges uniformly in \( i \), and, for suitable chosen \( N \),

\[
\sum_{n=N}^{\infty} |h_{n0}(i)|
\]

is bounded. By virtue of condition (iii), for fixed \( k \), we get that

\[
\sum_{n=0}^{\infty} |g_{nk}(i)|
\]

converges uniformly in \( i \). Since

\[
h_{nk}(i) = h_{n0}(i) - \sum_{k=0}^{k-1} g_{nk}(i),
\]

(2.7)
it follows that, for fixed $k$,
\[
\sum_{n=0}^{\infty} |h_{nk}(i)|
\] (2.8)
converges uniformly in $i$.

Now
\[
\Phi(i, Ax) = \sum_{k=0}^{\infty} g_{nk}(i) x_k
\]
\[
= \sum_{k=0}^{\infty} [h_{nk}(i) - h_{n,k+1}(i)] x_k
\]
\[
= \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}) ,
\] (2.9)
by (2.4) and the boundedness of $x_k$.

Condition (ii) and the boundedness of (2.6) show that
\[
\sum_{n=N}^{\infty} |h_{nk}(i)|
\] (2.10)
is bounded for all $k,i$. We can make
\[
\sum_{k=k_0+1}^{\infty} |x_k - x_{k-1}|
\]
arbitrarily small by choosing $k_0$ sufficiently large. It therefore follows that, given $\varepsilon > 0$, we can choose $k_0$ so that, for all $i$,
\[
\sum_{n=N}^{\infty} |h_{nk}(i)(x_k - x_{k-1})| < \varepsilon .
\] (2.11)
By the uniform convergence of (2.8), it follows that, once $k_0$ has been chosen, we can choose $n_0$ so that, for all $i$,
\[
\sum_{n=n_0+1}^{\infty} |h_{nk}(i)(x_k - x_{k-1})| < \varepsilon .
\]
It follows from (2.11) that the same inequality holds when $\sum_{n=0}^{\infty}$ is replaced by $\sum_{n=n_0+1}^{\infty}$; hence, for all $i$,
\[
\sum_{n=n_0+1}^{\infty} |h_{nk}(i)(x_k - x_{k-1})| < 2\varepsilon .
\] (2.12)
Hence,
\[
\sum_{n=n_0+1}^{\infty} \Phi_n(i, Ax) < 2\varepsilon .
\]
Thus
\[
\sum_{n=0}^{\infty} |\Phi_n(i, Ax)|
\]
converges uniformly.

Now, by virtue of (2.9), we have

\[
\lim_{n \to \infty} t_n(i, Ax) = t_{N-1}(i, Ax) = \sum_{n=N}^{\infty} \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}) = \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{n=N}^{\infty} h_{nk}(i)
\]

the assertion being justified by absolute convergence because of the boundedness of (2.10). By (2.7), we have

\[
\sum_{n=N}^{\infty} h_{nk}(i) = \sum_{n=N}^{\infty} h_{no}(i) - \sum_{\ell=0}^{k-1} \sum_{n=N}^{\infty} g_{n\ell}(i) = \alpha - \sum_{\ell=0}^{k-1} \alpha_{\ell} - \sum_{\ell=k}^{\infty} \sum_{p=0}^{\infty} b_{N-1,p}(i) a_{p\ell}.
\]

Thus,

\[
\lim_{n \to \infty} t_n(i, Ax) = \alpha \lim_{k \to \infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k \to \infty} x_k) \alpha_k.
\]

This completes the proof.

REFERENCES


