SOME REMARKS ON THE SPACE $R^2(E)$

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ABSTRACT. Let $E$ be a compact subset of the complex plane. We denote by $R(E)$ the algebra consisting of the rational functions with poles off $E$. The closure of $R(E)$ in $L^p(E)$, $1 \leq p < \infty$, is denoted by $R^p(E)$. In this paper we consider the case $p = 2$. In section 2 we introduce the notion of weak bounded point evaluation of order $\beta$ and identify the existence of a weak bounded point evaluation of order $\beta$, $\beta > 1$, as a necessary and sufficient condition for $R^2(E) \neq L^2(E)$. We also construct a compact set $E$ such that $R^2(E)$ has an isolated bounded point evaluation. In section 3 we examine the smoothness properties of functions in $R^2(E)$ at those points which admit bounded point evaluations.

KEY WORDS AND PHRASES. Rational functions, compact set, $L^p$-spaces, bounded point evaluation, weak bounded point evaluation, Bessel capacity.

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1. INTRODUCTION.

Let $E$ be a compact subset of the complex plane $\mathbb{C}$. For each $p$, $1 \leq p < \infty$, let $L^p(E)$ be the linear space of all complex valued functions $f$ for which $|f|^p$ is integrable with the usual norm

$$\left\{ \int_E |f(z)|^p \, dm(z) \right\}^{1/p},$$

where $m$ denotes the two dimensional Lebesgue measure. Denote by $R(E)$ the subspace of all rational functions having no poles on $E$ and let $R^p(E)$ be the closure of $R(E)$ in $L^p(E)$. A point $z_0 \in E$ is said to be a bounded point evaluation (BPE) for $R^p(E)$, if there is a constant $F$ such that
\[ |f(z_0)| \leq F \left( \int_E |f(z)|^p \, dm(z) \right)^{1/p}, \text{ for all } f \in \mathbb{R}^p(E). \] (1.1)

In [1] Brennan showed that \( \mathbb{R}^p(E) = L^q(E) \), \( p \neq 2 \), if and only if no point of \( E \) is a BPE for \( \mathbb{R}^p(E) \). The theorem is not true for \( p = 2 \) (See Fernström [2] or Fernström and Polking [3].) In this paper we show that if the right hand side of (1) is made slightly larger a corresponding theorem is true for \( p = 2 \). We also show that this theorem is best possible.

If \( z_0 \in E \) is a BPE for \( \mathbb{R}^p(E) \) there is a function \( g \in L^q(E) \), \( \frac{1}{p} + \frac{1}{q} = 1 \), such that
\[
|f(z_0)| = \int_E f(z)g(z) \, dm(z) \quad \text{for all } f \in \mathbb{R}^p(E).
\]
The function \( g \) is called a representing function for \( z_0 \). Let \( B(z, \delta) \) denote the ball with radius \( \delta \) and centre at \( z \). We say that a set \( A, A \subset \mathbb{R} \), has full area density at \( z \) if \( m(A \cap B(z, \delta)) / m(B(z, \delta)) \) tends to one when \( \delta \) tends to zero.

Suppose now that \( z_0 \) is a BPE for \( \mathbb{R}^p(E), 2 < p \), represented by \( g \in L^q(E) \) and \( (z - z_0)^{-s} \phi(|z - z_0|)^{-1} g \in L^q(E) \), where \( s \) is a nonnegative integer and \( \phi \) is a non-decreasing function such that \( r \phi(r)^{-1} \rightarrow 0 \) when \( r \rightarrow 0 \). Then for every \( \varepsilon > 0 \) there is a set \( E_0 \) in \( E \) having full area density at \( z_0 \) such that for every \( f \in \mathbb{R}^p(E) \) and for all \( \tau \in E_0 \),
\[
|f(\tau) - f(z_0)| \leq \left( \int_E |f(z)|^p \, dm(z) \right)^{1/p}.
\]
This theorem is due to Wolf [8].

We shall show that the theorem of Wolf is not true for \( p = 2 \). We shall also show that a slightly weaker result is true and that this result is best possible.

The main tool to show this is to construct a compact set \( E \) with exactly one bounded point derivation for \( \mathbb{R}^2(E) \). A point \( z_0 \in E \) is a bounded point derivation (BPD) of order \( s \) for \( \mathbb{R}^p(E) \) if the map \( f \mapsto f^{(s)}(z_0) \), \( f \in \mathbb{R}^p(E) \), extends from \( \mathbb{R}^p(E) \) to a bounded linear functional on \( \mathbb{R}^p(E) \).

2. BPE'S AND APPROXIMATION IN THE MEAN BY RATIONAL FUNCTIONS.

Denote the Bessel kernel of order one by \( G \) where \( G \) is defined in terms of its Fourier transform by
\[
\hat{G}(z) = (1 + |z|^2)^{-\frac{1}{2}}.
\]

For \( f \in L^2(\mathbb{C}) \) we define the potential
\[
u^f(z) = \int G(z-\tau) f(\tau) \, d\mu(\tau).
\]

The Bessel capacity \( C_2 \) for an arbitrary set \( X, X \subseteq \mathbb{C} \), is defined by
\[
C_2(X) = \inf \int |f(z)|^2 \, d\mu(z),
\]

where the infimum is taken over all \( f \in L^2(\mathbb{C}) \) such that \( f(z) \geq 0 \) and \( \nu^f(z) \geq 1 \) for all \( z \in X \). The set function \( C_2 \) is subadditive, increasing, translation invariant and
\[
C_2(B(z,\delta)) = (\log \frac{1}{\delta})^{-1},
\]

for \( \delta \leq \delta_0 < 1 \).

For further details about this capacity see Meyers [5].

The BPD's can be described by the Bessel capacity. Let \( A_n(z_0) \) denote the annulus
\[
\{ z; 2^{-n-1} < |z-z_0| < 2^{-n} \}.
\]

The following theorem is proved in [3]:

**Theorem 2.1** Let \( E \) be a compact set. Then \( z \) is a BPD of order \( s \) for \( R^2(E) \) if and only if
\[
\sum_{n=0}^{\infty} 2^{n(s+1)} C_2(A_n(z) - E) < \infty.
\]

**Definition** Set
\[
L_{z_0}(z) = \begin{cases} 
\log \frac{1}{|z-z_0|} & \text{for } |z-z_0| \leq \frac{1}{e} \\
1 & \text{for } |z-z_0| \geq \frac{1}{e}.
\end{cases}
\]

A point \( z_0 \in E \) is called a weak bounded point evaluation (w BPE) of order \( \beta, \beta > 0 \), for \( R^2(E) \) if there is a constant \( F \) such that
\[
|f(z_0)| \leq F \left\{ \int_E |f(z)|^2 L_{z_0}^\beta(z) \, d\mu(z) \right\}^{\frac{1}{2}}
\]

for all \( f \in R(E) \).

We are now going to generalize theorem 2.1 in two directions.

**Theorem 2.2** Let \( s \) be a nonnegative integer and \( E \) a compact set. Suppose that \( z_0 \) is a BPE for \( R^2(E) \) represented by \( g \in L^2(E) \) and that \( \phi \) is a positive, nondecreasing function defined on \((0,\infty)\) such that \( r \phi(r)^{-1} \) is nondecreasing and tends to zero when \( r \to 0^+ \). Then \( z_0 \) is represented by a function \( g \in L^2(E) \) such that
\[
\frac{g}{(z-z_0)^s \phi(|z-z_0|)} \in L^2(E).
\]
if and only if
\[ \sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(z^{-n})^{2} C_{2}(A_{n}(z^{0}) - E) < \infty. \]

Theorem 2.3 Let \( E \) be a compact set. Then \( z \) is a \( w \) BPE of order \( \beta \) for \( R^{2}(E) \) if and only if
\[ \sum_{n=1}^{\infty} 2^{2n(s+1)} \beta^{2} C_{2}(A_{n}(z) - E) < \infty. \]

The proofs of theorem 2.2 and theorem 2.3 are almost the same as the proof of theorem 2.1. We omit the proofs. Wolf proved in [8] that the condition
\[ \sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(z^{-n})^{2} C_{2}(A_{n}(z^{0}) - E) < \infty \]
is necessary in theorem 2.2.

The compact sets \( E \) for which \( R^{2}(E) = L^{2}(E) \) can be described in terms of the Bessel Capacity. The following theorem is proved in Hedberg [4] and Polking [6].

Theorem 2.4 Let \( E \) be a compact set. Then the following are equivalent.

(i) \( R^{2}(E) = L^{2}(E) \).

(ii) \( C_{2}(B(z,\delta) - E) = C_{2}(B(z,\delta)) \) for all balls \( B(z,\delta) \).

(iii) \( \lim_{\delta \to 0} \sup_{\delta} \frac{C_{2}(B(z,\delta) - E)}{\delta^{2}} > 0 \) for all \( z \).

If we combine theorem 2.3 and theorem 2.4 we get the following theorem.

Theorem 2.5 Let \( \beta > 1 \) and \( E \) be a compact set. Then \( L^{2}(E) = R^{2}(E) \) if and only if \( E \) admits no \( w \) BPE of order \( \beta \) for \( R^{2}(E) \).

Now we shall show that theorem 2.5 is not true for \( \beta \leq 1 \). We first need the following theorem.

Theorem 2.6 There is a compact set \( E \) such that

(i) \( C_{2}(B(0,\frac{1}{2}) - E) < C_{2}(B(0,\frac{1}{2})) \)

(ii) \( \sum_{n=1}^{\infty} n^{-1} 2^{2n} C_{2}(A_{n}(z) - E) = \infty \) for all \( z \).

The proof is a modification of a proof in [2] or [3], where a weaker theorem is proved. Since we shall need the construction of \( E \) later, we give some details.

Proof. There are constants \( F_{1} \) and \( F_{2} \) such that
\[ F_{1}(\log \frac{1}{\delta})^{-1} \leq C_{2}(B(z,\delta)) \leq F_{2}(\log \frac{1}{\delta})^{-1} \]
for all \( \delta, \delta \leq \delta_{0} < 1 \).
Choose \( a, a > 1 \), such that
\[
\frac{F_2}{a} \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < C_2(B(0,1/2)).
\]

Let \( A_0 \) be the closed unit square with centre at the origin. Cover \( A_0 \) with \( 4^n \) squares with side \( 2^{-n} \). Call the squares \( A_n(i), i = 1, 2, \ldots, 4^n \). In every \( A_n(i) \) put an open disc \( B_n(i) \) such that \( B_n(i) \) and \( A_n(i) \) have the same centre and the radius of \( B_n(i) \) is \( \exp(-a4^n n \log n) \). Repeat the construction for all \( n, n \geq 2 \).

Set
\[
E = A_0 - \bigcup_{n=2}^{\infty} \bigcup_{i=1}^{4^n} B_n(i).
\]

The subadditivity of \( C_2 \) now gives (i).

In order to prove (ii) it is enough to prove
\[
C_2(A_n(i) - E) \geq \frac{F_1}{32a4^n \log n} \quad \text{for all } n, n \geq n_0.
\]

Consider all \( B_k(i), n \leq k \leq n^2 \), such that \( B_k(i) \subset A_n(i) \).

We get \( 4^k \) discs with radius \( \exp(-a4^{n+k}(n+i) \log^2(n+i)) \), \( 0 < k < n^2 - n \).

Call the discs
\[
D_n(r), r = 1, 2, \ldots, \frac{4^{n^2-n+1} - 1}{3}.
\]

Thus
\[
\frac{F_1}{a4^n} \sum_{j=1}^{n^2} \frac{1}{j \log^2 j} \leq \sum_{i} C_2(D_n(r)) \leq \frac{F_2}{a4^n} \sum_{j=1}^{n^2} \frac{1}{j \log^2 j}.
\]

Set \( D_n = \bigcup_r D_n(r) \).

Since the distances between the discs are large compared to their radii, it can be shown that
\[
C_2(D_n) > \frac{1}{8} \sum_r C_2(D_n(r)), \text{ if } n \text{ is large.}
\]

(See theorem 2' in [2] or theorem 2 in [3] for a proof.)

Thus if \( n \) is large,
\[
C_2(A_n(i) - E) \geq C_2(D_n) \geq \frac{F_1}{8a4^n} \sum_{j=1}^{n^2} \frac{1}{j \log^2 j} \geq \frac{F_1}{16a4^n \log n},
\]

which is (2.1) q.e.d.
Theorem 2.7 There is a compact set \( E \) such that

(i) \( L^2(E) = R^2(E) \)

(ii) \( E \) has no \( \omega \) BPE of order one for \( R^2(E) \).

Proof The theorem follows immediately from theorem 2.3, 2.4, and 2.6.

3. BPE's AND SMOOTHNESS PROPERTIES OF FUNCTIONS IN \( R^2(E) \).

In this section we treat the theorem of Wolf mentioned in the introduction for the case \( p = 2 \).

Theorem 3.1 Let \( \phi \) be a positive, nondecreasing function defined on \( (0,\infty) \) such that

\[ r L_0(r) \phi(r)^{-1} \text{ is nondecreasing and } r L_0(r) \phi(r)^{-1} \rightarrow 0 \text{ when } r \rightarrow 0^+ . \]

Suppose that \( z_0 \) is a BPE for \( R^2(E) \) represented by \( g \) and

\[ (z-z_0)^{s} \phi(|z-z_0|)^{-1} g \in L^2(E), \text{ where } s \text{ is a nonnegative integer}. \]

Then for every \( s > 1 \) and \( \varepsilon > 0 \) there is a set \( E_0 \) in \( E \), having full area density at \( z_0 \), such that for every \( f \in R(E) \) and every \( \tau \in E_0 \)

\[ f(\tau) - f(z_0) = \frac{f'(z_0)}{1!} (\tau-z_0) - \frac{f''(z_0)}{2!} (\tau-z_0)^2 + \cdots + \frac{f^{(s)}(z_0)}{s!} (\tau-z_0)^s \]

\[ \leq \varepsilon |\tau-z_0|^{s} \phi(|\tau-z_0|) \left\{ \int_E |f(z)|^2 \ dm(z) \right\}^{\frac{1}{2}} . \]

The proof of theorem 3.1 is only a minor modification of the proof of theorem 4.1 in [8]. Moreover, there is a proof of theorem 3.1 for \( \beta = 2 \) in Wolf [7]. We omit the proof.

Remark. Let \( z_0 \in \partial E \) (the boundary of \( E \)) be both a BPE for \( R^2(E) \) and the vertex of a sector contained in \( \text{Int } E \). Let \( L \) be a line which passes through \( z_0 \) and bisects the sector. Let \( \varepsilon > 0 \) and let \( \phi \) be as in theorem 2.2. For those \( y \in L \cap E \) that are sufficiently near \( z_0 \) Wolf showed in [9] that

\[ |f(y)-f(z_0)| \leq \varepsilon \phi(|y-z_0|) \left\{ \int_E |f(z)|^2 \ dm(z) \right\}^{\frac{1}{2}} \quad \text{for all } f \in R(E). \]

Our next step is to prove that theorem 3.1 is not true for \( \beta = 1 \). We first need a theorem, which we think is interesting in itself.

Theorem 3.2 Let \( s \) be a nonnegative integer. Then there is a compact set \( E \) such that

(i) \( \sum_{n=1}^{\infty} n^{-1} 2^{2n} C_2(A_n(z) \cap E) = \infty \text{ if } z \neq 0 \).

(ii) \( \sum_{n=1}^{\infty} 2^{2n(s+1)} C_2(A_n(0) \cap E) < \infty \).
Proof We shall modify the set constructed in the proof of theorem 2.6. Let \( B_j^{(k)} \) denote the same discs as in that proof. Let all \( B_j^{(k)} \) which intersect \( A_1(0) \) be denoted by \( A_{11}, A_{12}, A_{13}, \ldots \) so that their diameters are decreasing.

Choose \( j_1 \) so that
\[
2^{2(s+1)} \sum_{j > j_1} C(A_{1j}) < 2^{-1}
\]
and
\[
\text{diam}(A_{1j_1}) < 2^{-3}.
\]
Suppose that we have chosen \( j_1, \ldots, j_n \). Let all \( B_j^{(k)} \) which intersect \( A_{n+1}(0) \) and which do not coincide with \( A_{11}, \ldots, A_{1j_1}, \ldots, A_{nj_n} \), be denoted by \( A_{n+1}, A_{n+2}, A_{n+3}, \ldots \) so that their diameters are decreasing.

Choose \( j_{n+1} \) so that
\[
2^{2(n+1)(s+1)} \sum_{j > j_{n+1}} C(A_{n+1j}) < 2^{-(n+3)}
\]
and
\[
\text{diam}(A_{n+1j_{n+1}}) < 2^{-(n+3)}.
\]

Let \( A_0 \) be the closed unit square with centre at the origin. Set \( E = A_0 - \{ \text{the union of all } B_j^{(k)} \text{ such that } B_j^{(k)} \not\subseteq A_{nm}, 1 \leq n < \infty \text{ and } 1 \leq m \leq j_n \} \).

We have
\[
\sum_{n=1}^{\infty} 2^{2n(s+1)} C(A_n(0) - E) \leq \sum_{n=1}^{\infty} 2^{-n} \leq \infty.
\]

Let \( z \not\in 0 \). If \( k \) is large all \( B_j^{(k)} \), \( B_j^{(k)} \subseteq A_k(z) \), differ from \( A_{nm} \), \( 1 \leq n < \infty \) and \( 1 \leq m \leq j_n \).

Now exactly as in proof of theorem 2.6 it follows
\[
\sum_{n=1}^{\infty} n^{-1} 2^{2n} C(A_n(z) - E_1) = \infty
\]
q.e.d.

Corollary 3.3 There is a compact set \( E \) with exactly one BPD of order \( s \) for \( \mathbb{R}^2(E) \).

Proof Just combine theorem 3.2 and 2.1.

Remark The situation for \( p \neq 2 \) is different. In [1] Brennan showed that if almost all points \( z \in E \), \( E \) compact, are not BPE for \( \mathbb{R}^p(E) \), \( E \) admits no BPE's for \( \mathbb{R}^2(E) \).

Theorem 3.4 Let \( s \) be a nonnegative integer and \( \phi \) be as in theorem 2.2. Then there is a compact set \( E \) such that

(i) \( z_0 \) is a BPE for \( \mathbb{R}^2(E) \).

(ii) There is a representing function \( g \) for \( z_0 \) that satisfies
(iii) For every $\tau \in E$, $\tau \neq z_0$, and every positive integer $n$ there is a function $f \in \mathcal{R}(E)$ such that

$$\left| f(\tau) - f(z_0) - \frac{f'(z_0)}{1!}(\tau - z_0) - \ldots - \frac{f^{(s)}(z_0)}{s!}(\tau - z_0)^s \right| > n^2 \left\{ \frac{1}{E} \left[ \left| f(z) \right|^2 L_{\mathbb{R}^d}(z) \right]^{\frac{1}{2}} \right\}$$

Proof Theorem 3.2 gives that there is a compact set $E$ such that

$$\sum_{n=1}^{\infty} n^{-1} z^{2n} C_2(A_n(z) - E) = \infty, z \neq z_0$$

$$\sum_{n=1}^{\infty} z^{2n(s+1)} \phi(2^{-n})^{-2} C_2(A_n(z_0) - E) < \infty.$$ 

Now theorem 2.1 gives (i) and theorem 2.2 gives (ii). Moreover theorem 2.1 gives that $z_0$ is a BPD of order $s$ for $\mathcal{R}^2(E)$ and theorem 2.3 that $\tau$ is not a w BPE of order 1 for $\mathcal{R}^2(E)$. This gives (iii).

REFERENCES


