RESEARCH NOTES

A CHARACTERIZATION OF THE DESARGUESIAN PLANES
OF ORDER $q^2$ BY $SL(2,q)$

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ABSTRACT. The main result is that if the translation complement of a translation plane of order $q^2$ contains a group isomorphic to $SL(2,q)$ and if the subgroups of order $q$ are elations (shears), then the plane is Desarguesian. This generalizes earlier work of Walker, who assumed that the kernel of the plane contained $GF(q)$.

KEY WORDS AND PHRASES. Translation planes, translation complement, elations.


THEOREM. Let $\pi$ be a translation plane of order $q^2$, where $q = p^r$ and $p$ is a prime. Let $G \cong SL(2,q)$ be a subgroup of the translation complement of $\pi$ whose elements of order $p$ are elations. Then $\pi$ is a Desarguesian plane.

This theorem is a special case required in the classification of all translation planes $\pi$ of order $q^2$ which admit a collineation group $G \cong SL(2,q)$ [1, 2]. That classification is a generalization of the work of Walker and Schaeffer [3, 4], who assume, in addition, that the kernel of $\pi$ contains $GF(q)$.

To begin the proof, let $W$ be a vector space of dimension $2r$ over $GF(p)$. Since
π is a 4r-dimensional vector space over GF(p), we may represent π as W ⊗ W so that the points of π are vectors (x,y), where x,y ∈ W. The components of π (i.e., the lines containing (0,0)) have the form [(0,y): y ∈ W] and [(x,xA): x ∈ W] for various GF(p)—linear transformations A: W → W. We will denote the components by their defining equations x = 0 and y = xA, respectively. Next, note that each Sylow p-subgroup Q of G is abelian and hence all the elements (≠ 1) of Q have the same elation axis. Let S denote the set of all components of π and let N be the subset of elation axes; thus |S| = q^2 + 1 and |N| = q + 1.

**Lemma 1.** (Hering [5], Ostrom [6]). We may coordinatize π as above such that

\[ G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D ∈ K ; \ AD - BC = I \right\} \]

where K is a field of 2r x 2r matrices over GF(p) and K ≅ GF(q). Further, the elation axes (that is, the elements of N) have the form y = xA (A ∈ K) and x = 0.

**Lemma 2.** There is an element g ∈ G such that the following conditions are satisfied: (i) |g| = q + 1; (ii) |g|^p t - 1 for t < 2r; and (iii) g fixes a component of π which is not in the set N.

**Proof.** The integer s is a p-primitive prime divisor of q^2 - 1 if s is a prime, s|q^2 - 1, and s|p^t - 1 for 0 < t < 2r (hence s|q + 1). q^2 - 1 has a p-primitive prime divisor s unless q = 8 or q = p and p + 1 = 2^a [7]. In the first case, let |g| = s so that g satisfies conditions (i) and (ii). Then g also satisfies condition (iii) because |g| is a prime and g permutes the q(q-1) components in S\N. If q = 8, choose g such that |g| = 9. Since |S\N| = 56 ≠ 0 (mod 3), g must fix one of the elements of S\N. Finally, if q = p and p + 1 = 2^a, choose h of order 8 in G and let g = h^2. Then g^2 has order 2 in G = SL(2,K), so g^2 = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} fixes every component of π. Hence, h has orbits of lengths 1, 2, and 4 in S, and since 4|p(p-1) then h has an orbit of length 1 or 2 on S\N. Therefore g = h^2 fixes an element of S\N.

**Lemma 3.** Choose g ∈ G so that g satisfies the conditions of Lemma 2, and let L(g) be the ring of matrices generated by g over GF(p). Then L(g) is a field \( GF(q^2) \) and L(g) contains the subfield

\[ K = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A ∈ K \right\}. \]

**Proof.** g ∈ G ⊂ GL(2,K) by Lemma 1. As a 2 x 2 matrix over K, g has a minimum
polynomial \( f(x) \) over \( K \) of degree \( \leq 2 \). Since \( |g| \mid q(q-1) \), then the degree of \( f \) is 2 and \( f \) is irreducible over \( K \). Therefore, \( g \) and \( K \) generate a field \( U \cong GF(q^2) \) which contains \( L(g) \) as a subfield. Since \( |g| \mid p^t - 1 \) (for \( t < 2r \)), then \( L(g) = U \) and \( L(g) \supseteq \bar{K} \).

**Lemma 4.** Let \( g \) of Lemma 2 fix the component \( y = xT \) of \( S \setminus N \). Then \( K[T] \) is a field isomorphic to \( GF(q^2) \).

**Proof.** \( L(g) \) and hence \( \bar{K} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in K \right\} \) fix the component \( y = xT \), and thus \( K \) centralizes \( T \). \( T \) and the elements of \( K \) are \( 2r \times 2r \) matrices which act on a vector space \( V = V(2r,p) \) of dimension \( 2r \) over \( GF(p) \). \( K \) makes \( V \) into a 2-dimensional vector space and \( T \) acts as a \( K \)-linear transformation of \( V \). Hence, the minimum polynomial \( f(x) \) of \( T \) over \( K \) has degree \( \leq 2 \). If \( T \) has an eigenvalue \( A \) in \( K \), then the distinct components \( y = xT \) and \( y = xA \) of \( \pi \) must intersect, which is impossible. Therefore, \( T \) is irreducible over \( K \) and \( K[T] \cong GF(q^2) \).

We can now complete the proof of the Theorem. Let \( \pi^* \) denote the Desarguesian affine plane of order \( q^2 \) coordinatized by the field \( L = K(T) \); i.e., the points of \( \pi^* \) are \( \{(x,y) : x,y \in L\} \) and the components of \( \pi^* \) are \( \{y = xC : C \in L\} \cup \{x = 0\} \). Clearly, \( GL(2,L) \) acts as a collineation group of \( \pi^* \). We superimpose \( \pi^* \) on \( \pi \) by identifying the points of \( \pi^* \) and \( \pi \). Since \( K \subseteq L \) and \( T \in L \), the components \( y = xA \) of \( N \) and \( y = xT \) are components both of \( \pi^* \) and \( \pi \). Since \( G = SL(2,K) \subseteq GL(2,L) \), then \( G \) acts both as a collineation group of \( \pi^* \) and of \( \pi \). Finally, recall that \( SL(2,K) \) acts transitively on the \( q(q-1) \) components of \( \pi^* \) outside of \( N \) (for example, the stabilizer subgroup in \( SL(2,K) \) of a component of \( \pi^* \) outside \( N \) has order \( q+1 \)). Therefore, the images of \( y = xT \) under \( G \) constitute \( q(q-1) \) components both of \( \pi^* \) and of \( \pi \); so \( \pi^* = \pi \) as required.

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**REFERENCES**


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