A NOTE ON A PAPER BY S. HABER

A. McD. MERCER

Department of Mathematics and Statistics
University of Guelph
Guelph, Ontario, Canada N1G 2W1

(Received September 23, 1982 and in revised form February 26, 1983)

ABSTRACT. A technique used by S. Haber to prove an elementary inequality is applied here to obtain a more general inequality for convex sequences.

KEY WORDS AND PHRASES. Convex sequences, Hadamard’s inequality for convex functions, rearrangements.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 26D15, Secondary 40G05.

1. INTRODUCTION.

Let a and b be non-negative. Then the following elementary inequality was proved in [1].

\[ \frac{1}{n+1} \left[ a^n + a^{n-1} b + \ldots + b^n \right] \geq \left( \frac{a+b}{2} \right)^n \quad (n=0, 1, 2, \ldots) \ldots (1.1) \]

Now this inequality can be obtained at once by taking \( f(t) = t^n \) in the well-known result

\[ \frac{1}{b-a} \int_a^b f(t) dt \geq f\left( \frac{a+b}{2} \right) \quad \ldots (1.2) \]

which holds whenever \( f \) is convex in \([a, b]\). However, the method used in [1] to obtain (1.1) is interesting and it is the purpose of the present note to show that it can be used to prove a considerably more general result about sequences. Indeed this more general result will have (1.2) as a consequence.

2. MAIN RESULTS.

A lemma which we shall use is the following

LEMMA. If

\[ \beta_0 \geq \beta_1 \geq \beta_2 \geq \ldots \geq \beta_m \]

and

\[ \sum_{v=0}^{m} \alpha_v = 0 \]

and
and if the ordering of the $a_v$ is such that each positive $a$ precedes all the negative ones, then

$$m \sum_{v=0} a_v b_v \geq 0.$$ 

This lemma, which is easily proved, is not the one stated by Haber but, essentially, it is what he used. For with $b_i$ defined as in [1]

$$(i = 0, 1, 2, \ldots, \left[\frac{n}{2}\right]: n \text{ even})$$

we do not in fact have

$$\left[\frac{n}{2}\right] \sum_{i=0} b_i = 0$$

which is what is needed to apply the lemma quoted there.

Our result is the following.

**THEOREM.** Let $\{u_v\}_{v=0}^n$ be a convex sequence. Then

$$\frac{1}{n+1} \sum_{v=0}^n u_v \geq \frac{1}{2^n} \sum_{v=0}^n \left(\begin{array}{c} n \\ v \end{array}\right) u_v \ .... (2.1)$$

To see that (1.2) is a consequence of (2.1) let the function $f(x)$ be bounded and convex (and hence continuous) on $[a, b]$ and take

$$u_v = f(a + \frac{v}{n}(b-a)).$$

Then (2.1) reads

$$\frac{1}{n+1} \sum_{v=0}^n f(a + \frac{v}{n}(b-a)) \geq \frac{1}{2^n} \sum_{v=0}^n \left(\begin{array}{c} n \\ v \end{array}\right) f(a + \frac{v}{n}(b-a)) \ .... (2.2)$$

On letting $n \to \infty$ the left-hand side here tends to the left-hand side of (1.2). And by virtue of Bernstein's result

$$\lim_{n \to \infty} \sum_{v=0}^n \left(\begin{array}{c} n \\ v \end{array}\right) \phi \left(\frac{v}{n}\right)(1-x)^{(1-x)n-v} = \phi(x) \ .... (2.3)$$

whenever $\phi \in C[0,1]$ we see that the right-hand side of (2.2) tends to $f(\frac{a+b}{2})$.

Merely take $\phi(x) = f(a + x(b-a))$ and $x = 1/2$ in (2.3).

We now proceed to prove (2.1).

**PROOF.** Following Haber let us put $Q = \left[\frac{n}{2}\right]$ and write

$$\sum_{v=0}^Q \gamma_v = \begin{cases} \gamma_0 + \gamma_1 + \ldots \ldots \ldots + \gamma_Q \text{ if } n \text{ is odd} \\ \gamma_0 + \gamma_1 + \ldots + \gamma_{Q-1} + \frac{1}{2} \gamma_Q \text{ if } n \text{ is even} \end{cases}$$

Then

$$\frac{1}{n+1} \sum_{v=0}^n u_v - \frac{1}{2^n} \sum_{v=0}^n \left(\begin{array}{c} n \\ v \end{array}\right) u_v = \sum_{v=0}^Q \gamma_v [u_v + u_{n-v}]$$
where

\[ c_v = \frac{1}{n+1} - \frac{1}{2R(v)} \]

Since \( \{u_v\}_{v=0}^n \) is convex then

\[ u_{v+1} + u_{n-v-1} \leq u_v + u_{n-v} \quad (0 \leq v \leq n-1) \]

which is to say that the sequence \( \{u_v + u_{n-v}\}_{v=0}^Q \) is non-increasing. We see too that the sequence \( \{c_v\}_{v=0}^Q \) is non-increasing and that \( \sum_{v=0}^Q c_v = 0 \). Appealing to the Lemma quoted above we find that

\[ \sum_{v=0}^Q c_v [u_v + u_{n-v}] \geq 0 \]

and this completes the proof of (2.1).

In conclusion I wish to thank the referee for his helpful advice concerning the lemma used here.

REFERENCES
