ON THE RADIUS OF UNIVALENCE OF
CONVEX COMBINATIONS OF ANALYTIC FUNCTIONS

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ABSTRACT. We consider for $\alpha > 0$, the convex combinations $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$, where $F$ belongs to different subclasses of univalent functions and find the radius for which $f$ is in the same class.

KEY WORDS AND PHRASES. Univalent functions, alpha-quasi-convex, starlike, close-to-convex functions, convex combinations.

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1. INTRODUCTION.

Let $S$, $K$, $S^*$ and $C$ denote the classes of analytic functions in the unit disc $E = \{z: |z| < 1\}$ which are respectively univalent, close-to-convex, starlike, and convex. In [1,2], a new subclass $C^*$ of univalent functions was introduced and studied. A function $f$, analytic in $E$, belongs to $C^*$ if and only if there exists a convex function $g$ such that for $z \in E$,

$$\text{Re} \left( \frac{zf'(z)}{g'(z)} \right) > 0. \quad (1.1)$$

The functions in $C^*$ are called quasi-convex and $C \subset C^* \subset K \subset S$. It is shown [2] that $f \in C^*$ if and only if $zf' \in K$. Recently the functions called $\alpha$-quasi-convex have been defined and their properties studied in [3]. A function $f$, analytic in $E$, is said to be $\alpha$-quasi-convex if and only if there exists a convex function $g$ such that, for $\alpha$ real and positive

$$\text{Re} \left( (1 - \alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{zf'(z)}{g'(z)} \right) > 0. \quad (1.2)$$
It has been shown [3] that $F$ is $\alpha$-quasi-convex if and only if $f$ with
\[ f(z) = (1 - \alpha)F(z) + \alpha zF'(z) \] is close-to-convex. (1.3)

All $\alpha$-quasi-convex functions are close-to-convex.

2. MAIN RESULTS.

We shall now study the mapping properties of $f$: $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$, \( \alpha > 0 \), when $F$ belongs to different subclasses of univalent functions.

THEOREM 2.1. Let $F \in S^*$ and $\alpha > 0$. The function
\[ F(z) = (1 - \alpha)F(z) + \alpha zF'(z) \] (2.1)
is starlike in $|z| < r_0$, where
\[ r_0 = \frac{1}{2\alpha + \sqrt{4\alpha^2 + 1 - 2\alpha}}. \] (2.2)
This result is sharp.

PROOF. We can write (2.1) as
\[ f(z) = \alpha z \frac{1}{\alpha} \left( z^{\frac{1}{\alpha}} F(z) \right)' , \]
and from this it follows that
\[ F(z) = \frac{1}{\alpha} \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz. \] (2.3)

Then
\[ \frac{2F'(z)}{F(z)} = \left\{(1 - \frac{1}{\alpha}) \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz + f(z))/\left\{ \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz \right\} \right\} \]
\[ = \left\{(1 - \frac{1}{\alpha}) \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz + \frac{1}{\alpha} - 1 \right\} \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz \}
\[ = h(z), \] (2.4)
where $\text{Re } h(z) > 0$, since $F \in S^*$.

From (2.4), we have
\[ \frac{1}{z^{\alpha}} f(z) - \left( \frac{1}{\alpha} - 1 \right) \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz = h(z) \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz. \] (2.5)

Differentiating both sides of (2.5), we obtain
\[ \left( \frac{1}{\alpha} - 1 \right) z^{\alpha} f(z) + z^{\alpha} f'(z) - \left( \frac{1}{\alpha} - 1 \right) z^{\alpha} f(z) = h'(z) \int_0^z z^{\alpha} f(z)dz + h(z) z^{\alpha} f(z). \]

Thus
\[ \frac{z f'(z)}{f(z)} = h(z) + \left\{ h'(z) \int_0^z z^{\frac{1}{\alpha} - 2} f(z)dz \right\} \frac{1}{f(z)}. \]
Now, using the well-known result [4], \( |h'(z)| \leq \frac{2 \text{Re } h(z)}{1 - r^2} \), \( |z| = r \), we have

\[
\text{Re} \frac{zf'(z)}{f(z)} \geq \text{Re} h(z) \left\{ 1 - \frac{2}{1 - r^2} \left[ \int_0^z \frac{\frac{1}{z} - 2}{\frac{1}{z} - 2 f(z)dz} \right] \right\}. \tag{2.6}
\]

From (2.1) and (2.3), we have

\[
\int_0^z \frac{\frac{1}{z} - 2}{\frac{1}{z} - 2 f(z)dz} \frac{1}{\alpha z^{\frac{1}{\alpha}} - 1 f(z)} = \frac{1}{\alpha z^{\frac{1}{\alpha}} - 1 F(z)} \frac{1}{\alpha z^{\frac{1}{\alpha}} - 1 F(z)} = \frac{1}{\alpha z^{\frac{1}{\alpha}} - 1 F(z)} \frac{1}{\alpha z^{\frac{1}{\alpha}} - 1 F(z)} = \frac{zF'(z)}{F(z)} + \left( \frac{1}{\alpha} - 1 \right) = h(z) + \left( \frac{1}{\alpha} - 1 \right),
\]

from which it follows that

\[
\left| \frac{\frac{1}{\alpha} - 1 f(z)}{\int_0^z \frac{\frac{1}{z} - 2}{\frac{1}{z} - 2 f(z)dz}} \right| \geq \text{Re} \left\{ h(z) + \left( \frac{1}{\alpha} - 1 \right) \right\} \geq \left( \frac{1}{\alpha} - 1 \right) + \frac{1 - r}{1 + r}. \tag{2.7}
\]

Using (2.7), we have from (2.6)

\[
\text{Re} \frac{zf'(z)}{f(z)} \geq \text{Re} h(z) \left\{ 1 - \left( \frac{2}{1 - r^2} \right) \left( 1 + \frac{r^2}{\frac{1}{\alpha} + (\frac{1}{\alpha} - 2)r} \right) \right\}
\]

\[
= \text{Re} h(z) \left\{ \frac{1}{\alpha} - 4r - (\frac{1}{\alpha} - 2)r^2 \right\}/(1 - r)(\frac{1}{\alpha} + (\frac{1}{\alpha} - 2)r). \tag{2.8}
\]

The right hand side of (2.8) is positive for \( r < r_o \), where \( r_o \) is given by (2.2). This result is sharp as can be seen by

\[
f_o(z) = \{\alpha(\frac{1}{\alpha} - 2z)\}/(1 - z)^3
\]

\[
= (1 - \alpha)F_0(z) + \alpha zF_0'(z), \tag{2.9}
\]

where

\[
F_0(z) = \frac{z}{(1 - z)^2} \in S^*.
\]

REMARK 2.1. Let \( f \in C \), then \( f \), given by (2.1), is convex for \( |z| < r_o \), where \( r_o \) is given by (2.2). The proof follows on the same lines as in Theorem 2.1. See also [5] and [6].

REMARK 2.2. In [6], Nikolaeva and Repnina treated the same problem, with a different notation, for the convex and starlike functions of order \( \beta \). Theorem 2.1 follows from their result when we take \( \beta = 0 \) for \( 0 \leq \alpha \leq 1 \). On the other hand, our proof of Theorem 2.1 is much simpler and the result holds for all \( \alpha > 0 \).
THEOREM 2.2. Let \( F \in K \) and \( f(z) = (1 - \alpha)F(z) + \alpha zF'(z), \alpha > 0 \). Then \( f \) is close-to-convex in \( |z| < r_0 \), \( r_0 \) is given by (2.2). The function \( f_0 \) in (2.9) shows that this result is sharp.

PROOF. Since \( F \in K \), there exists a \( G \in S^* \) such that, for \( z \in E \), \( \Re zF'(z) \). Now let \( g(z) = (1 - \alpha)G(z) + \alpha zG'(z) \). Then by Theorem 2.1, \( g \) is starlike for \( |z| < r_0 \), \( r_0 \) is defined by (2.2). Using the same technique of Theorem 2.1, we can easily show that \( \Re \frac{zf'(z)}{g(z)} > 0 \) for \( |z| < r_0 \).

REMARK 2.3. For \( \alpha = \frac{1}{2} \), this result has been proved in [7].

As an easy consequence of (1.3) and Theorem 2.2, we have the following.

COROLLARY 2.1. Let \( F \in K \) and \( f(z) = (1 - \alpha)F(z) + \alpha zF'(z), \alpha > 0 \). Then \( F \) is \( \alpha \)-quasi-convex in \( |z| < r_0 \). This means that the radius of \( \alpha \)-quasi-convexity for close-to-convex functions is given by (2.2).

THEOREM 2.3. Let \( F \in C^* \) and \( \alpha > 0 \). Let \( f(z) = (1 - \alpha)F(z) + \alpha zF'(z). \) Then \( f \) is in \( C^* \), for \( |z| < r_0 \), \( r_0 \) is given by (2.2).

PROOF. Since \( F \in C^* \), there exists a \( G \in C \) such that for \( z \in E \), \( \Re (zF'(z))' \). Now let \( g(z) = (1 - \alpha)G(z) + \alpha zG'(z) \), then \( g \) is convex in \( |z| < r_0 \). We can write
\[
\begin{align*}
f(z) &= (1 - \alpha)F(z) + \alpha zF'(z) = z \left( 2 - \frac{1}{\alpha} \right) \left( \frac{1}{\alpha} - 1 \right) \left( F(z) \right) \\
g(z) &= (1 - \alpha)G(z) + \alpha zG'(z) = z \left( 2 - \frac{1}{\alpha} \right) \left( \frac{1}{\alpha} - 1 \right) \left( G(z) \right).
\end{align*}
\]
Thus
\[
\frac{(zf'(z))'}{g'(z)} = \frac{2 - \frac{1}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \left( F(z) \right)'}{(\alpha - 1) \left( G(z) \right)'}.
\]
Now
\[
\begin{align*}
(z(z \left( 2 - \frac{1}{\alpha} \right) \left( \frac{1}{\alpha} - 1 \right) \left( F(z) \right)')')' &= (z(2 - \alpha - 1)F(z) + zF'(z))' = (2 - \alpha) zF'(z) + z F''(z))' \\
&= \left( 2 - \frac{1}{\alpha} \right) \left( 2 - \alpha - 1 \right) \left( z \left( \frac{1}{\alpha} \right) F'(z) + z \alpha F''(z) \right)')' = \left( 2 - \frac{1}{\alpha} \right) \left( 2 - \alpha - 1 \right) \left( zF'(z) \right)'
\end{align*}
\]
Let \( zF'(z) = H(z) \), then from (2.10), we have
\[
\frac{(zf'(z))'}{g'(z)} = \frac{2 - \frac{1}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \left( z \left( \frac{1}{\alpha} \right) H(z) \right)'}{(\alpha - 1) \left( z \left( \frac{1}{\alpha} \right) G(z) \right)'}.
\]
Since from Theorem 2.2, the function \( (1 - \alpha)H(z) + zH'(z) = z \left( 2 - \frac{1}{\alpha} \right) \left( \frac{1}{\alpha} - 1 \right) \left( z \left( \frac{1}{\alpha} \right) H(z) \right) \) belongs to \( K \) with respect to a convex function \( g: g(z) = (1 - \alpha)G(z) + \alpha zG'(z) \) in
\[ |z| < r_o , \text{ so } f \text{ is in } C^* \text{ for } |z| < r_o , \text{ where } r_o \text{ is given by (2.2).} \]

REMARK 2.4. For \( F \in C^* \) and \( \alpha = \frac{1}{2} \), Theorem 2.3 has been proved in [1].

We now deal with a generalized form of (1.1) by taking \( g \) to be starlike and prove the following.

THEOREM 2.4. Let \( F \) be analytic in \( E \) and let for \( z \in E, \text{ Re } G'(z) > 0, G \in S^* \).

Let \( f(z) = (1 - \alpha)F(z) + azF'(z) \) and \( g(z) = (1 - \alpha)G(z) + azG'(z) \), with \( \alpha > 0 \). Then

\[ \text{Re} \left( \frac{zf'(z)}{g'(z)} \right)^* > 0 \text{ for } |z| < r_1, \text{ where} \]

\[ r_1 = \frac{1}{3\alpha + \sqrt{9\alpha^2 + 1 - 2\alpha}} \]

For \( \alpha = \frac{1}{2} \), the problem has been solved in [8].

PROOF. From (2.3), we can write

\[ F(z) = \frac{1}{\alpha} \int_0^z \frac{1}{z^{\alpha - 2}} f(z) \, dz \]

\[ zF'(z) = \frac{1}{\alpha} \int_0^z \frac{1}{z^{\alpha - 2}} f(z) \, dz + \frac{1}{\alpha} f(z) \]

Thus

\[ (ZF'(z))^* = \frac{1}{(z^\alpha f'(z)) - (\frac{1}{\alpha} - 1) \int_0^z \frac{1}{z^{\alpha - 1}} f'(z) \, dz} = h(z), \quad (2.11) \]

where \( \text{Re } h(z) > 0, \ z \in E. \)

From (2.11), we write

\[ \frac{1}{z^\alpha f'(z)} = (\frac{1}{\alpha} - 1) \int_0^z \frac{1}{z^{\alpha - 1}} f'(z) \, dz = h(z) \int_0^z \frac{1}{z^{\alpha - 1}} g'(z) \, dz. \]

Differentiating both sides, and simplifying, we obtain

\[ \frac{(zf'(z))^*}{g'(z)} = h(z) + \frac{h'(z) \left( \int_0^z \frac{1}{z^{\alpha - 1}} g'(z) \, dz \right)}{z^{\alpha - 1} g'(z)} \quad \cdot \quad (2.12) \]

Using \( |h'(z)| \leq \frac{2\text{Re } h(z)}{1 - r^2} \), (2.12) gives
\[
\Re \left( \frac{zf'(z)}{g'(z)} \right) \geq \Re h(z) \left[ 1 - \frac{2}{1 - r^2} \left| \int_0^z \frac{1}{z^\alpha} \frac{1}{g'(z)dz} \right| \right]
\]

(2.13)

Now
\[
\frac{1}{z^\alpha g'(z)/( \int_0^z \frac{1}{z^\alpha} g'(z)dz)} = \frac{(1/\alpha)G'(z) + zG''(z)}{G'(z)} = \frac{1}{\alpha} - 1 + \frac{(zG'(z))'}{G'(z)}.
\]

(2.14)

Since \( G \in S^* \), so
\[
\left| \frac{(zG'(z))'}{G'(z)} \right| \geq \frac{1 - 4r + r^2}{1 - r^2}.
\]

(2.15)

From (2.13), (2.14) and (2.15), we obtain
\[
\Re \left( \frac{zf'(z)}{g'(z)} \right) \geq \Re h(z) \left[ 1 - \frac{2}{1 - r^2} \frac{r(1 - r^2)}{\frac{1}{\alpha} - 4r - (\frac{1}{\alpha} - 2)r^2} \right]
\]
\[
= \Re h(z) \frac{1 - 6ar - (1 - 2a)r^2}{1 - 4ar - (1 - 2a)r^2},
\]

and this positive for \( |z| < r_1 \), where
\[
r_1 = \frac{1}{3a + \sqrt{9a^2 + 1 - 2a}}.
\]

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