A HILLE-WINTNER TYPE COMPARISON THEOREM FOR SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. For the linear difference equation

$$\Delta(c_{n-1} \Delta x_{n-1}) + a_n x_n = 0 \quad \text{with} \quad c_n > 0,$$

a non-oscillation comparison theorem given in terms of the coefficients $c_n$ and the series $\sum_{n=k}^{\infty} a_n$, has been proved.

KEY WORDS AND PHRASES. Difference equations, oscillatory and non-oscillatory solutions, comparison theorems, Riccati transformation.


1. INTRODUCTION.

We consider linear homogeneous second order difference equations of the form

$$\Delta(c_{n-1} \Delta x_{n-1}) + a_n x_n = 0, \quad n = 1, 2, 3, \ldots,$$

(1.1)

where $\Delta$ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$, and $a = \{a_n\}$ and $c = \{c_n\}$ are real-valued infinite sequences with $c_n > 0$ for $n = 0, 1, 2, \ldots$. (No assumption is made about the sign of $a_n$.)

Equation (1.1) is equivalent to the difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n,$$

(1.2)

where $b_n = c_n + c_{n-1} - a_n$, $n = 1, 2, 3, \ldots$. Recent papers ([1], [2], and [3]) have
treated oscillation and comparison theorems for this equation.

The theorem to be considered here is a difference equation analogue of Taam's
generalized version [4] of the well-known Hille-Wintner comparison theorem for
second-order linear differential equations (see [5, Thm. 7, p. 245] and [6], or
[7, p. 60-62]).

Let \( x = \{x_n\} \), \( n = 0,1,2,\ldots \), be a real, non-trivial solution of (1). Then \( x \)
is said to be oscillatory if, for every \( N \), there exists \( n_0 \geq N \) such that
\( x_n x_{n+1} \leq 0 \). Since either all non-trivial real solutions of (1.1) are oscillatory
or none are (see [8, p. 153]), equation (1.1) may be classified as oscillatory or
non-oscillatory. Also, if \( x \) is a solution of (1.1), so is \( -x \), and it then follows
that (1.1) is non-oscillatory if and only if there exists a solution \( x \) with \( x_n > 0 \)
for all \( n \geq N \), for some integer \( N \geq 0 \). (The variables \( j, k, n, M, N \) will always
be understood below to represent non-negative integers.)

2. MAIN RESULTS.

We will prove the following comparison result:

**THEOREM 1.** Given the difference equations

\[
\Delta(c_{n-1} \Delta x_{n-1}) + A_n x_n = 0 \\
\Delta(c_{n-1} \Delta x_{n-1}) + a_n x_n = 0,
\]

(2.1)

(2.2)

assume that

\[
0 < c_n \leq c_n \quad \text{and} \quad c_n \leq K
\]

(2.3)

for all \( n \geq 0 \), for some constant \( K > 0 \), and

\[
0 \leq \sum_{n=k}^{\infty} a_n \leq \sum_{n=k}^{\infty} A_n < \infty
\]

(2.4)

for all sufficiently large \( k \). Then, if (2.1) is non-oscillatory, (2.2) is non-
oscillatory also.

Before proceeding to the proof of Theorem 1, we need two preliminary results.
The first of these is an elementary property of real numbers:

**LEMMA 1.** If \( 0 \leq a \leq b \) and \( c > 0 \), then
\[ 0 \leq \frac{a^2}{a + c} \leq \frac{b^2}{b + c}. \]

Our second lemma may be thought of as a discrete analogue of Theorem 4 of Hille [5, p. 243], which gives a necessary and sufficient condition for non-oscillation of solutions of \( x'' + f(t)x = 0 \) in terms of the existence of a solution of a related Riccati integral equation. Hille then used a successive approximations technique to show the existence of a solution of the integral equation. We will use a similar device here to prove Theorem 1. Our proof of Lemma 2 is a discrete version of a standard Riccati transformation argument, as used, for example, in the proof of the Hille-Wintner theorem presented by Swanson [7]. The resulting difference equation (2.7) below is quite different in form, however, from the Riccati differential equation.

**Lemma 2.** Assume that
\[ \sum_{n=1}^{\infty} A_n < \infty. \]

Then the difference equation (1.2) is non-oscillatory if and only if there exists a sequence \( v \) satisfying
\[ v_k = \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + C_n} + \sum_{n=k}^{\infty} A_{n+1} \]
for all sufficiently large \( k \), say \( k \geq M \), with \( v_k + C_k > 0 \) for \( k \geq M \).

**Proof.** Let (2.1) be non-oscillatory and let \( x \) be a solution of (2.1) with \( x_n > 0 \) for \( n \geq M \). Let \( v_n = C_n (\Delta x_n)/x_n \), \( n \geq M \). Then
\[ v_n = C_n \left( \frac{x_{n+1}}{x_n} - 1 \right) = C_n \left( \frac{x_{n+1}}{x_n} - 1 \right) > -C_n, \]
so \( v_n + C_n > 0 \), \( n \geq M \). From (2.1) we have
\[ C_{n+1} \Delta x_{n+1} - C_n \Delta x_n + A_{n+1} x_{n+1} = 0. \]
Dividing by \( x_{n+1} \) and adding and subtracting \( v_n \), one obtains
\[ v_{n+1} - v_n + v_n - v_n x_n/x_{n+1} + A_{n+1} = 0, \]

or

\[ \Delta v_n + v_n (1 - x_n/x_{n+1}) + A_{n+1} = 0. \]

(2.6)

Now

\[ \frac{x_{n+1}}{x_n} = \frac{x_{n+1} - x_n + x_n}{x_n} = \frac{\Delta x_n}{x_n} + 1 = \frac{v_n}{C_n} + 1, \]

and substitution of this expression into (2.6) yields

\[ \Delta v_n + \frac{v_n^2}{v_n + C_n} + A_{n+1} = 0, \quad n \geq M. \]

(2.7)

Summing from \( k \) to \( N \), where \( M \leq k < N \), we obtain

\[ v_{N+1} - v_k + \sum_{n=k}^{N} \frac{v_n^2}{v_n + C_n} = - \sum_{n=k}^{N} A_{n+1}. \]

(2.8)

By hypothesis, the right side of (2.8) has a finite limit as \( N \to \infty \), so the left side also has such a limit.

As noted above, \( v_n + C_n > 0 \) for \( n \geq M \), so \( v_n^2/(v_n + C_n) \geq 0 \) for \( n \geq M \). If

\[ \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + C_n} = +\infty, \]

then, from (2.8), \( v_{N+1} \to -\infty \) as \( N \to \infty \). But this is impossible, since \( v_n \geq -C_n \), and \( -C_n \geq -K \) by hypothesis. Thus, for every \( k \geq M \), we have

\[ 0 \leq \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + C_n} < \infty. \]

Therefore, \( v_n^2/(v_n + C_n) \to 0 \) as \( n \to \infty \), from which it follows, since \( C_n \) is bounded, that \( v_n \to 0 \) as \( n \to \infty \). Equation (2.5) then follows immediately from (2.8).

Conversely, if \( v \) is a sequence satisfying (2.5), with \( v_n + C_n > 0 \) for \( n \geq M \), then application of the forward difference operator \( \Delta \) to both sides of (2.5) leads immediately to equation (2.7). We then define a sequence \( x \) inductively as

\[ x_M = 1, \quad x_{n+1} = \left( \frac{v_n + C_n}{C_n} \right)x_n, \quad n \geq M. \]

Then \( x_n > 0 \) for \( n \geq M \), and \( v_n = C_n (x_{n+1}/x_n - 1) \); hence, \( v_n = C_n \Delta x_n/x_n \). Substitution of this expression into equation (2.7) then leads readily to equation (2.1),
so \( x_n \) as defined above satisfies (2.1) for \( n \geq M \). We may then define
\( x_{M-1}, x_{M-2}, \ldots, x_0 \) successively, using (2.1). The resulting sequence \( x \) is thus a non-oscillatory solution of (2.1), which completes the proof of Lemma 2.

Proceeding with the proof of Theorem 1, we assume that (2.1) is non-oscillatory. Then by Lemma 2 there exists a sequence \( V = \{v_k\}, \ k \geq M \), for some \( M \geq 0 \), which satisfies (2.5), with \( V_k + C_k > 0 \) for \( k \geq M \). We will use a successive approximations argument to show that there is a sequence \( v = \{v_k\}, \ k \geq M \), which satisfies

\[
 v_k = \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + c_n} + \sum_{n=k}^{\infty} a_{n+1}, \quad k \geq M. \tag{2.9}
\]

It will then follow by Lemma 2 that equation (2.2) is non-oscillatory.

We define a sequence of successive approximations

\[
v^j = \{v^j_k\}, \ k \geq M, \ j \geq 0, \text{ as follows:} \]

\[
v_k^0 = v_k, \quad k \geq M, \tag{2.10a}
\]

\[
v_k^j = \sum_{n=k}^{\infty} \frac{(v_{k-1}^j)^2}{v_n + c_n} + \sum_{n=k}^{\infty} a_{n+1}, \quad k \geq M, \ j \geq 1. \tag{2.10b}
\]

We must first show that the sequences \( v^j \), \( j \geq 1 \), are well-defined by (2.10ab).

Since \( v_k^0 = v_k, \ k \geq M \), we have

\[
v_k^0 + c_k > v_k^0 + C_k = V_k + C_k > 0, \quad k \geq M.
\]

Then

\[
 0 \leq \frac{(v_k^0)^2}{v_k^0 + c_k} \leq \frac{(v_k^0)^2}{v_k^0 + C_k} = \frac{v_k^2}{V_k + C_k}, \quad k \geq M. \tag{2.11}
\]

Since the sequence \( V \) satisfies (2.5), it follows from (2.11) that the series in (2.10ab) converges for \( j = 1 \), for all \( k \geq M \). Therefore \( v_k^1 \) is well-defined by (2.10ab) for each \( k \geq M \). Furthermore, by (2.4), (2.10ab), and (2.11), we have

\[
 0 \leq v_k^1 \leq \sum_{n=k}^{\infty} \frac{(v_n)^2}{v_n + c_n} + \sum_{n=k}^{\infty} A_{n+1} = V_k = v_k^0, \quad k \geq M;
\]

i.e.,

\[
 0 \leq v_k^1 \leq v_k^0, \quad k \geq M.
\]
Proceeding inductively, we assume that $v_k^j$, $k \geq M$, has been defined by (2.10ab) for $j = 1, 2, 3, \ldots, i$ and that $0 \leq v_k^i \leq v_k^{i-1}$ for $k \geq M$, $j = 1, 2, 3, \ldots, i$. Then

$$v_k^i + c_k \geq c_k > 0, \ k \geq M. \quad \text{Using Lemma 1, we then obtain}$$

$$0 \leq \sum_{n=k}^{\infty} \frac{(v_n^i)^2}{v_n^i + c_n} \leq \sum_{n=k}^{\infty} \frac{(v_n^{i-1})^2}{v_n^{i-1} + c_n}, \ k \geq M. \quad (2.12)$$

It follows from (2.12) that $v_k^{i+1}$ is well-defined by (2.10ab) for all $k \geq M$, and from (2.10ab) and (2.12) we have

$$0 \leq v_k^{i+1} \leq v_k^i, \ k \geq M.$$

Therefore, by induction, $v_k^j$ is defined by (2.10ab) for all $j \geq 1$ and $k \geq M$, and

$$0 \leq v_k^j \leq v_k^{j-1}, \ j \geq 1, \ k \geq M. \quad (2.13)$$

Thus $v_k^j$ is non-negative and non-increasing in $j$ for each $k \geq M$ and we may define

$$v_k = \lim_{j \to \infty} v_k^j, \ k \geq M.$$

Note that $v_k \geq 0$, so that $v_k + c_k \geq c_k > 0, \ k \geq M$. Also, from (2.13) and Lemma 1, we obtain

$$0 \leq \frac{(v_k^j)^2}{v_k^j + c_k} \leq \frac{(v_k^{j-1})^2}{v_k^{j-1} + c_k}, \ j \geq 1, \ k \geq M. \quad (2.14)$$

Repeated application of (2.14) yields, with (2.13),

$$0 \leq \frac{(v_k^j)^2}{v_k^j + c_k} \leq \frac{(v_k^j)^2}{v_k^j + c_k} \quad \text{for all} \ j \geq 0.$$

Thus the convergence of the first series in (2.10b) is uniform with respect to $j$. Consequently, we may take limits in (2.10ab) as $j \to \infty$ and obtain equation (2.9). It then follows from Lemma 2 that equation (2.2) is non-oscillatory, which completes the proof of the theorem.

Since several recent discussions of oscillation of solutions of difference equations have treated equation (1.2) above, we restate theorem 1 for equations of the form (1.2).
THEOREM 2. Given the difference equations

\[ C_n x_{n+1} + C_{n-1} x_{n-1} = B_n x_n \]  \hspace{1cm} (2.15)

\[ C_n x_{n+1} + C_{n-1} x_{n-1} = b_n x_n \]  \hspace{1cm} (2.16)

assume that \( 0 < C_n \leq c_n \) and \( C_n \leq K, n = 0,1,2, \ldots \), for some constant \( K > 0 \). If

\[ 0 \leq \sum_{n=k}^{\infty} (c_n + c_{n-1} - b_n) \leq \sum_{n=k}^{\infty} (C_n + C_{n-1} - B_n) < \infty \]

for all sufficiently large \( k \), then, if (2.15) is non-oscillatory, (2.16) is non-oscillatory also.

REFERENCES


