FUNCTIONS IN THE SPACE $R^2(E)$
AT BOUNDARY POINTS OF THE INTERIOR

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ABSTRACT. Let $E$ be a compact subset of the complex plane $\mathbb{C}$. We denote by $R(E)$ the algebra consisting of (the restrictions to $E$ of) rational functions with poles off $E$. Let $m$ denote 2-dimensional Lebesgue measure. For $p \geq 1$, let $R^p(E)$ be the closure of $R(E)$ in $L^p(E, dm)$.

In this paper we consider the case $p = 2$. Let $x \in \partial E$ be a bounded point evaluation for $R^2(E)$. Suppose there is a $C > 0$ such that $x$ is a limit point of the set $S = \{y \in \text{Int} E, \text{Dist}(y, \partial E) \geq C \cdot |y - x|\}$. For those $y \in S$ sufficiently near $x$ we prove statements about $|f(y) - f(x)|$ for all $f \in R(E)$.

KEY WORDS AND PHRASES. Rational functions, compact set $L^p$ spaces, bounded point evaluation, admissible function.


1. INTRODUCTION AND DEFINITIONS

Let $E$ be a compact subset of the complex plane $\mathbb{C}$. We denote by $R(E)$ the algebra consisting of (the restrictions to $E$ of) rational functions with poles off $E$. Let $m$ denote 2-dimensional Lebesgue measure. For $p \geq 1$, let $R^p(E)$ be the closure of $R(E)$ in $L^p(E, dm)$. A point $x \in E$ is said to be a bounded point evaluation (BPE) for $R^p(E)$ if there is a constant $F$ such that

$$\frac{1}{P} \int_E |f(z)|^p dm(z) \cdot |f(x)| \leq F$$

for all $f \in R(E)$. 

In [4] we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at BPE's. When $p = 2$, the situation is quite different (see Fernström and Polking [2] and Fernström [1]). In [5] we showed that at certain BPE's the functions in $R^2(E)$ have the following smoothness property: Let $x \in \partial E$ be both a BPE for $R^2(E)$ and the vertex of a sector contained in Int $E$. Let $L$ be a line segment that bisects the sector and has an end point at $x$. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $y \in L$ and $|y - x| < \delta$, $|f(y) - f(x)| \leq \varepsilon \|f\|_2$ for all $f \in R(E)$. The goal of this paper is to extend this result to certain cases where there may not be a sector in Int $E$ having vertex at $x$, but $x$ is still a limit point of Int $E$.

If $x \in E$ is a BPE for $R^2(E)$, there is a function $g \in L^2(E)$ such that $f(x) = \int_E fg \, dm$ for any $f \in R(E)$. Such a function $g$ is called a representing function for $x$.

A point $x \in E$ is a bounded point derivation (BPD) of order $s$ for $R^2(E)$ if the map $f \mapsto f^{(s)}(x)$, $f \in R(E)$, extends from $R(E)$ to a bounded linear functional on $R^2(E)$.

Let $A_n(x)$ denote the annulus $\{z|2^{-n-1} \leq |z - x| \leq 2^{-n}\}$. Let $A_n'(x) = \{z|2^{-n-2} < |z - x| < 2^{-n+1}\}$. If $x = 0$, we will denote $A_n(0)$ by $A_n$ and $A_n'(0)$ by $A_n'$.

For an arbitrary set $X \subset \mathbb{C}$ we let $C_2(X)$ denote the Bessel capacity of $X$ which is defined using the Bessel kernel of order 1 (see [3]).

We say that $\phi$ is an admissible function if $\phi$ is a positive, non-decreasing function defined on $(0, \infty)$, and $r \cdot \phi(r)^{-1}$ is nondecreasing and tends to zero when $r \to 0^+$.

Using the techniques of [4] and [2] one can prove:

**Theorem 1.1.** Let $s$ be a nonnegative integer and $E$ a compact set. Suppose that $x$ is a BPE for $R^2(E)$ and $\phi$ is admissible. Then $x$ is
represented by a function $g \in L^2(E)$ such that

$$g \cdot \phi(|z-x|) \in L^2(E)$$

if and only if $\sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(2^{-n})^{-2} C_2 (A_n(x) - E) < \infty$.

2. **THE MAIN RESULTS**

Let $x \in \partial E$ be a BPE for $R^2(E)$. We may assume that $x = 0$ and that $E \subset \{ |z| < 1 \}$. Suppose there is a positive constant $C$ such that $0$ is a limit point of the set $S = \{ y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C |y| \}$.

We will construct a function $g \in L^2(E)$ which represents $0$ for $R^2(E)$ and has support disjoint from $S$.

**LEMMA 2.1.** Let $0 \in \partial E$ be a BPE for $R^2(E)$. Suppose there is a positive constant $C$ such that $0$ is a limit point of the set $S = \{ y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C |y| \}$. Then there is a function $g \in L^2(E)$ such that:

(i) $g$ represents $0$ for $R^2(E)$,

(ii) $m(\text{supp } g) \cap S) = 0$,

$$k = \left\lceil \frac{n-2}{2} \right\rceil + 1$$

(iii) For all $n \geq 2$, \[ \int_{A_n \cap E} |g|^2 \, dm \leq F \sum_{k=0}^{2^{n-1}} C_2 (A_{2k+1} - E) \]

where $F$ is a constant independent of $n$.

**PROOF.** For each $i, i = 0, 1, 2, \ldots$ consider all the intersections of the set $A_i = \{ z | 2^{-i-1} \leq |z| \leq 2^{-i} \}$ with the bounded components of $E - E$. Let $Y_i$ be the closure of the union of these intersections.

Since $Y_i$ is compact, it can be covered by finitely many open discs of radius $<C 3^{-i-1}$. Let the union (finite) of these discs be denoted by $B_i$. The set $B_i$ is bounded by finitely many closed Jordan curves each of which is the union of finitely many circular arcs. Each set $B_i$ is contained in a set $C_i$ bounded by finitely many closed Jordan curves $\Gamma_{ij}$, $j = 1, 2, \ldots, n_i$ such that if $z$ belongs to any one of these
curves, \( \text{Dist} (z, B_1) = 2C_3^{-1}z^{-1} \).

Now for each \( k, k = 0, 1, 2, \ldots \) choose a function \( \lambda_k \in C^1_0 \) such that:

1. \( \supp \lambda_k \subset A_{2k+1} \)

2. \( \lambda_k(z) = 1 \) for \( z \in \{z | 2^{-2k-2} \leq |z| \leq 2^{-2k-1} \} \cap B_{2k+1} \)

3. \( \lambda_k(z) = 0 \) for \( z \in \bigcup_{i=0}^{\infty} C_i \)

4. \( \frac{\partial \lambda_k(z)}{\partial x_1} \leq F_1 \cdot 2^{2k+1}, \quad \frac{\partial \lambda_k(z)}{\partial x_2} \leq F_2 \cdot 2^{2k+1} \)

where \( z = x_1 + ix_2 \) and \( F_1 \) and \( F_2 \) are constants independent of \( k \).

5. \( \lambda_k(z) + \lambda_{k+1}(z) = 1 \) for \( z \in \{z | 2^{-2k-3} \leq |z| \leq 2^{-2k-2} \} \cap B_{2k+2} \).

Given any \( \varepsilon > 0 \) we use a lemma of Fernström and Polking [2] to obtain functions \( \psi_k \in C^\infty \) such that:

1. \( \psi_k(z) \equiv 1 \) for \( z \) near \( A_k' = [z | \text{Dist}(z, E) < \varepsilon] \).

2. \( \int |D^\beta \psi_k(z)|^2 \, dm(z) \leq F \cdot 2^{-2k(1-|\beta|)} C_2(A_k' - E) \)

for \( \beta = (0,0), (0,1), \) and \( (1,0) \). Here the constant \( F \) is independent of \( \varepsilon \) and \( k \).

Since \( \supp \lambda_k \subset A_{2k+1} \), we have \( \psi_{2k+1} \cdot \lambda_k = \lambda_k \) on the set \( \{z | \text{Dist}(z, E) \geq \varepsilon\} \). Thus \( \sum_{\beta}^{\infty} \psi_{2k+1} \cdot \lambda_k \equiv 1 \) on \( \{|z| \leq 4^{-1}\} \) - \( \{z | \text{Dist}(z, E) < \varepsilon\} \). Choose \( \chi \in C^\infty \) with \( \chi(z) \equiv 1 \) near \( E \). Set \( h(z) = \chi(z) \cdot \frac{1}{|z|^2} \). For each double index \( \beta = (0,0), (0,1), \) and \( (1,0) \) there is a constant \( F_{\beta} \) such that

\[
|D^\beta h(z)| \leq F_{\beta} \cdot |z|^{-1-|\beta|}.
\]

Set \( f_\varepsilon = h \cdot \sum_{\beta}^{\infty} \psi_{2k+1} \cdot \lambda_k = \sum_{\beta}^{\infty} \psi_{2k+1} \cdot h_k \)

where \( h_k = \lambda_k h \).
Since $\text{supp } \lambda_k \subset A_{2k+1}^1$, the above inequalities imply that
\[
|D^B h_k(z)| \leq F_{B^2}(2k+1)(1+|B|)
\]
The subadditivity of $C_2$ and the convergence of $\sum_0^\infty 2^{2k} C_2(A_k - E)$ (see Theorem 1.1) imply that the net $\{f_\varepsilon\}$ is bounded in $L^2_{1,\text{loc}}$. There is a subsequence that converges weakly to a function $f \in L^2_{1,\text{loc}}$ which satisfies $f(z) = \frac{\chi(z)}{nz}$ for $z \in \mathbb{C} - E$ and $f(z) = 0$ for every $z \in E \cap \{z|\text{Dist}(z, \partial E) \geq C|z|\}$. Set $g = -\frac{\partial f}{\partial z}$. Then $g \in L^2(E)$ since $f \in L^2_{1}(E)$, and $g$ is a representing function for $0$. The proof of (iii) proceeds as in [5].

The above lemma can be used to prove the following theorem in almost the same way that in [5] Lemma 5.1 is used to prove Theorem 5.1.

**THEOREM 2.1.** Let $0 \in \partial E$ be a BPE for $R^2(E)$. Let $C$ be a positive constant such that $0$ is a limit point of the set $S = \{y|y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$. Let $g$ be a representing function for $0$ and suppose that $g(z) \cdot \phi(|z|)^{-1} \in L^2(E)$ where $\phi$ is an admissible function. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,
\[
|f(y) - f(0)| \leq \varepsilon\phi(|y|)\|f\|_2
\]
for all $f \in R(E)$.

Using this theorem and the methods in [4] one can prove:

**COROLLARY 2.1.** Suppose that all the conditions of Theorem 2.1 hold. Suppose, moreover, that $s$ is a positive integer such that $g(z) \cdot z^{-s} \cdot \phi(|z|)^{-1} \in L^2(E)$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,
\[
|f(y) - f(0) - \frac{f'(0)}{1!} (y - 0) - \cdots - \frac{f^{(s)}(0)}{s!} (y - 0)^s| \leq \varepsilon |y-0|^s \phi(|y|)\|f\|_2
\]
for all $f \in R(E)$. 
Finally, there is a corollary with weaker preconditions.

**COROLLARY 2.2.** Let $0, g,$ and $\phi$ be as in Theorem 2.1. Suppose there is a positive constant $C$ such that $0$ is a limit point of the set

$$S = \{ y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C \cdot \phi(|y|)|y| \}$$

Then for each $\epsilon > 0$ there is a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0)| \leq \epsilon \|f\|_2 \text{ for all } f \in R(E).$$

The proof is similar to the proof of Theorem 2.1.

One uses the fact that there exists an admissible function $\overline{\phi}$ such that $g \cdot \overline{\phi}^{-1} \cdot \phi^{-1} \in L^2(E)$.

3. **EXAMPLES**

**EXAMPLE 1.** We will construct a compact set $E$ such that $0 \in \partial E$, $0$ is a BPE for $R^2(E)$, and $0$ is a limit point of $\text{Int } E$. Let $D = \{ |z| \leq 1 \}$. Let $D_i$, $i = 1, 2, 3, \ldots$, be the open disc centered on the positive real axis at $3 \cdot 2^{-i-3}$ and having radius $r_i = \exp(-2^{2i-1})$.

Let $E = D - \bigcup_{i=1}^{\infty} D_i$. Then since $C_2(B(x,r)) \leq F(\log \frac{1}{r})^{-1}, r \leq r_0 < 1$, (see [3]), we have

$$\sum_{n=1}^{\infty} 2^n C_2(A_n - E) = \sum_{n=1}^{\infty} 2^n C(D_n) \leq F \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$ 

Thus $0$ is a BPE for $R^2(E)$. If $C$ is a positive constant sufficiently small (any positive number $< \frac{1}{2}$ will do), the set $\{ y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C |y| \}$ intersects the positive real axis in a sequence of disjoint intervals $[a_n, b_n]$ such that $b_n \to 0$.

**EXAMPLE 2.** Next we construct a compact set $E$ which is like Example 1 in that $0$ is a limit point of $\text{Int } E$ and a BPE for $R^2(E)$. In this example, however, there exists no sequence $\{y_n\} \subset \text{Int } E$ such that $|f(y_n) - f(0)| \leq \epsilon \|f\|_2$ for all $f \in R(E)$ if $|y_n| < \delta$. We will use
important parts of Fernström's construction in [1]. Let $F$ be a positive constant such that $C_2(B(z,r)) \leq F(\log \frac{1}{r})^{-1}$ for all $r, r < r_0 < 1$. Choose $a, a \geq 1$ such that

$$
\frac{F}{a} \sum_{n=1}^{\infty} \frac{1}{n \log 2n} < C_2(B(0,1/2)).
$$

Let $A_0$ be the closed unit square with center at 0. Cover $A_0$ with $2^{-n}$ squares of side $2^{-n}$. Call the squares $A_n^{(i)}, i = 1, 2, \ldots, 4^n$. In every set $A_n^{(i)}$ put an open disc $B_n^{(i)}$ such that $B_n^{(i)}$ and $A_n^{(i)}$ have the same center, and the radius of $B_n^{(i)}$ is $\exp(-4^n n \log 2n)$. Let $D_i, i = 1, 2, 3, \ldots$ be an open disc centered on the positive real axis such that $D_i \subset \{z | 2^{-i-1} \leq |z| \leq 2^{-i}\}$ and $r_i = \exp(-2^{2i} i^2)$. For each $n, n = 1, 2, 3, \ldots$, let $G_n = \bigcup_{i=1}^{4^n} B_n^{(i)}$ where the summation is over those indices $i$ such that $1 \leq i \leq 4^n$ and $B_n^{(i)} \cap (\bigcup_{i=1}^{n} D_i) = \emptyset$. Set $E_1 = A_0 - \bigcup_{n=2}^{\infty} G_n$. Then $R^2(E_1)$ has no BPE's in $\partial E_1$ as is shown in [1].

Now replace a suitable number of the discs $B_n^{(i)}, B_n^{(i)} \subset \bigcup_{j=1}^{\infty} G_j$, to obtain a compact set $E_2$ such that 0 is the only boundary point of $E_2$ that is a BPE for $R^2(E_2)$, (see [1]). This can be done so that $\text{Int } E_2 = \bigcup_{i=1}^{\infty} D_i$. If $y \in \text{Int } E_2$, let $\text{norm}(y)$ denote the norm of "evaluation at $y$" as a linear functional on $R^2(E_2)$. Then if $[y_k] \subset D_i$, and $y_k \to \partial D_i$, $\text{norm}(y_k) \to \infty$; otherwise some point on $\partial D_i$ would be a BPE for $R^2(E_2)$.

For each $i$ choose an open disc $D_i' \subset D_i$ such that $D_i'$ and $D_i$ are concentric and such that if $y \in D_i - D_i'$, then $\text{norm}(y)$ for the space $R^2(E_2 - D_i')$ is greater than $i$. 
Now let \( E = E_2 - \bigcup_{i=1}^{\infty} D_i' \).

The radii of the \( D_i' \) are so small that 0 is also a BPE for \( R^2(E) \). Let \( \{y_n\} \) be any sequence in \( \text{Int} \ E \) such that \( y_n \to 0 \). Let \( \text{norm}(y_n) = \text{norm of "evaluation at } y_n \text{" on } R^2(E) \). Then for no \( \epsilon > 0 \) is there a \( \delta > 0 \) such that if \( |y_n| < \delta \), \( |f(y_n) - f(0)| \leq \epsilon \|f\|_2 \) for all \( f \in R(E) \).

**EXAMPLE 3.** Let \( \varphi \) be an admissible function. Obtain a compact set \( E \) in the same way that the set \( E_2 \) was obtained in Example 2 so that:

1. \( D_i \) is centered at \( 3 \cdot 2^{-i-2} \) and has radius
   \[ r_i = \varphi(3 \cdot 2^{-i-2}) \cdot 2^{-i-2} \]
2. \( \sum_{n=0}^{\infty} 2^n \cdot \varphi(2^{-n})^2 C_2(A_n(0) - E) < \infty \), and
3. \( \sum_{n=0}^{\infty} 2^n C_2(A_n(x) - E) = \infty \) for \( x \neq 0 \), \( x \not\in \bigcup D_i \).

Let \( y_i = 3 \cdot 2^{-i-2} \). Then by the choice of \( r_i \), \( \text{Dist}(y_i, E) \geq 3^{-1} \varphi(|y_i|) |y_i| \). But there is no \( C > 0 \) such that \( \text{Dist}(y_i, \partial E) \geq C|y_i| \) for all \( i \). Hence Corollary 2.2 applies to the sequence \( \{y_i\} \) but Theorem 2.2 does not.

**REFERENCES**

1. Fernström, C., Some remarks on the space \( R^2(E) \), Math. Reports, University of Stockholm 1982.