IDENTITIES INVOLVING ITERATED INTEGRAL TRANSFORMS

CYRIL NASIM

Department of Mathematics and Statistics
The University of Calgary
Calgary, Alberta Canada T2N 1N4

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ABSTRACT. A number of identities involving iterated integral transforms are established, making use of the fact that a function which is a linear combination of the Macdonald's function $K_v(z)$, where $z$ is a complex variable, is a Fourier kernel.

KEY WORDS AND PHRASES. Macdonald's function, Fourier kernel, Mellin transform, Hankel transform, Laplace transform.

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1. INTRODUCTION.

The object of this note is to establish various identities involving integral operators. The integral operators are the integral transforms with respect to the function $K_v(z)$, where $K_v(z)$ is the Macdonald's function of order $v$ and argument $z$, a complex variable. Some functional relations are deduced, as special cases, which show the inter-relations among more familiar Fourier Sine, Fourier Cosine, and Laplace transforms.

2. THE KERNEL.

Let

$$y = x^v K_v(\theta x),$$

with $\theta$ a constant and $|\nu| < 1$.

Then

$$y'' - \frac{v^2 - 1/4}{x^2} y = \theta^2 y$$

or

$$\left\{ p^2 - \frac{v^2 - 1/4}{x^2} \right\} y = \theta^2 y, \quad p \equiv \frac{d}{dx}.$$ (2)

Whence

$$\left\{ p^2 - \frac{v^2 - 1/4}{x^2} \right\}^n y = \theta^{2n} y, \quad n = 0, 1, 2, \ldots.$$ (3)

Now, if we set $\theta = i e^{im/k}, \quad 0 \leq m \leq 2k-1$, (*)
then
\[ y = x^{\frac{1}{2}}K_{\nu}(ie^{i\pi m/k}x), \]
satisfies a \( k \)-fold Bessel equation:
\[
\left( D^2 - \frac{\nu^2 - 1/4}{x^2} \right)^k y = (-1)^k y
\]  
(2.1)

It is not difficult to see that if \( x \) is a complex variable, then every point of (2.1) is regular except for a singularity at \( x = 0 \). Now consider a function of the form
\[
G_k(x) = \sum_{m=0}^{2k-1} B_m x^{\frac{1}{2}}K_{\nu}(ie^{i\pi m/k}x), \quad |\nu| < 1.
\]
The functions are an extension of the functions which were first noted by Guinand.

As a special case when \( k = 2 \), chose the coefficients as
\[
B_0 = B_2 = -\frac{1}{\pi}, \quad B_1 = 0 \quad \text{and} \quad B_3 = -\frac{2}{\pi} \cos \frac{1}{2} \nu \pi .
\]

Then we obtain
\[
G_2(x) = -\frac{1}{\pi} x^{\frac{1}{2}}(K_{\nu}(ix) + K_{\nu}(-ix) + 2\cos(\frac{1}{2} \nu \pi)K_{\nu}(x))
\]
\[
= K_{\nu}(x), \quad \text{say},
\]
and we have

**THEOREM 2.1.** \( y = k_{\nu}(x) \) is a solution of
\[
\left( D^2 - \frac{\nu^2 - 1/4}{x^2} \right)^2 y = y, \quad 0 < x < \infty,
\]
the two-fold Bessel equation.

The function \( k_{\nu}(x) \) is of special interest to us here and we shall develop its properties further.

Using the representations [3]
\[
K_{\nu}(x) = \frac{\pi}{2} \csc \nu \pi (I_{-\nu}(x) - I_{\nu}(x))
\]
and
\[
Y_{\nu}(x) = \csc \nu \pi \{\cos \nu \pi J_{\nu}(x) - J'_{\nu}(x)\},
\]
where \( J_{\nu}, I_{\nu} \) and \( Y_{\nu} \) are the usual Bessel functions, equation (2.2), can be written as
\[
k_{\nu}(x) = x^{\frac{1}{2}}\{\sin \frac{1}{2} \nu \pi J_{\nu}(x) + \cos \frac{1}{2} \nu \pi (Y_{\nu}(x) + \frac{2}{\pi} K_{\nu}(x))\}.
\]

These functions arise as kernels in divisor summation formulae of the Hardy-Landau
type, involving number theoretic function \( \sigma_k(n) \), the number of \( k \)th powers of the divisor of \( n \), [4]. If we put \( \nu = \pm \frac{1}{2} \), we have

\[
k_{\frac{1}{2}}(x) = \pi^{-\frac{1}{2}}(\cos x - \sin x + e^{-x^2}),
\]

which obviously satisfies the differential equation

\[D^k y = y.\]

Next, the Mellin transform of \( x^k x_k(x) \) is given by

\[
m(x^k x_k(x)) = \alpha^{-s-\frac{1}{2}} \Gamma\left(\frac{1}{2} s + \frac{1}{2} \nu + \frac{1}{4}\right) \Gamma\left(\frac{1}{2} s - \frac{1}{2} \nu + \frac{1}{4}\right),
\]

where \( \text{Re} s > |\text{Re} \nu| - \frac{1}{2} \), [5], whence the Mellin transform of \( k(x) \) defined in (2.2) is given by

\[
k^*_\nu(s) = -\frac{2^{\nu + 1}}{\pi} \Gamma\left(\frac{1}{2} s + \frac{1}{2} \nu + \frac{1}{4}\right) \Gamma\left(\frac{1}{2} s - \frac{1}{2} \nu + \frac{1}{4}\right)
\]

\[
\left(\lambda^{-s-\frac{1}{2}} + (-\lambda)^{-s-\frac{1}{2}} + 2\cos \frac{1}{2} \nu \pi\right).
\]

On simplifying, we have

\[
k^*_\nu(s) = -\frac{2^{\nu + 1}}{\pi} \Gamma\left(\frac{1}{2} s + \frac{1}{2} \nu + \frac{1}{4}\right) \Gamma\left(\frac{1}{2} s - \frac{1}{2} \nu + \frac{1}{4}\right)
\]

\[
\cos \frac{1}{4} \pi(s + \nu + \frac{1}{2}) \cdot \cos \frac{1}{4} \pi(s - \nu + \frac{1}{2}),
\]

for \( \text{Re} s > |\text{Re} \nu| - \frac{1}{2} \).

By repeated use of the relation

\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},
\]

it is not a difficult matter to see that

\[
k^*_\nu(s)k^*_\nu(1-s) = 1.
\]

Hence,

THEOREM 2.2. The function \( k^*_\nu(x) \) defined by equation (2.2) is a Fourier kernel, [6].

If we define the transformation \( T[f] \) by
\[ T[f] = \int_{0}^{\infty} k_{\nu}(xt)f(t)dt, \quad |\nu| < 1, \quad (2.3) \]

then \( T^2[f] = f \) and \( T \) is involutory since \( T^2 = I \), the identity transformation. Making use of the asymptotic expansions of the Macdonald's function \( K_{\nu}(z) \), we have

\[ k_{\nu}(x) = O(e^{-x^2}), \quad x \to \infty \]

and

\[ k_{\nu}(x) = O(x^{\nu+\frac{1}{2}}), \quad x \to 0. \]

Therefore \( k_{\nu}(x) \in L^2(0,\infty) \), for \(|\nu| < 1\).

Now, if we take \( f(x) \in L^2(0,\infty) \), the integral defining the transformation \( T \) exists and is in fact absolute convergent. Thus \( T \) is a bounded transformation on \( L^2 \)-space for \(|\nu| < 1\).

3. THE OPERATOR.

We shall now define the transformation \( T \) in operator notation. Denote the operators \( K_{\nu} \) and \( K_{\nu,\pm} \) respectively by

\[ K_{\nu}[f] \equiv K_{\nu}[f(x); x] = \int_{0}^{\infty} \sqrt{xt} K_{\nu}(xt)f(t)dt \]

and

\[ K_{\nu,\pm}[f] \equiv K_{\nu}[f(x); i\pm x] = \int_{0}^{\infty} \sqrt{xt} K_{\nu}(i\pm xt)f(t)dt, \]

where \( f \in L^2(0,\infty) \) and \(|\nu| < 1\), with \( K_{\nu}(z) \) being the Macdonald's function. Then the transformation \( T \) can be expressed, in operator form, as

\[ T[f] = \int_{0}^{\infty} k_{\nu}(xt)f(t)dt \]

\[ = -\frac{1}{\pi} \left\{ \int_{0}^{\infty} \sqrt{xt} K(i\pm xt)f(t)dt + \int_{0}^{\infty} \sqrt{xt} K(-i\pm xt)f(t)dt \right\} + 2\cos \frac{1}{2} \nu \pi \int_{0}^{\infty} \sqrt{xt} K_{\nu}(xt)f(t)dt \]

\[ = -\frac{1}{\pi}(K_{\nu,\pm} + K_{\nu,-\pm}) + 2\cos \frac{1}{2} \nu \pi K_{\nu}[f]. \]

and we can write, symbolically,

\[ T = -\frac{1}{\pi}(K_{\nu,\pm} + K_{\nu,-\pm}) + 2\cos \frac{1}{2} \nu \pi K_{\nu} \]
Since $T^2 = I$, the identity transformation, we have

$$\frac{1}{\pi^2} (K_{\nu, i} + K_{\nu, -i} + 2\cos \frac{1}{2} \nu \pi K_{\nu})^2 = I$$

or

$$I = \frac{1}{\pi^2} \left\{ K_{\nu, i}^2 + K_{\nu, -i}^2 + 4\cos^2 \frac{1}{2} \nu \pi K_{\nu} + K_{\nu, i} K_{\nu, -i} + K_{\nu, i} K_{\nu, -i} + K_{\nu, i} K_{\nu, -i} \right\}$$

(3.1)

The right-hand side is the linear combination of iterated transformations, which are bounded on $L^2$-space for $|\nu| < 1$.

Now, using the standard result [5],

$$\int_0^\infty t K_{\nu}(at) K_{\nu}(bt) dt = \frac{\pi(a/b)^{\nu}}{\sin \nu \pi} \left( \frac{a^2 - b^2 \nu}{\alpha^2 - \beta^2} \right),$$

where $|\nu| < 1$ and $\text{Re}(a+b) > 0$, we have for example, if $f' \in L^2(0, \infty)$,

$$K_{\nu, i} K_{\nu} [f'] = \frac{1}{\pi^2} \int_0^\infty e^{-xt} K_{\nu}(it) dt \int_0^\infty (ut)^{i-1/2} K_{\nu}(ut) f(u) du$$

$$= \frac{1}{\pi^2} \int_0^\infty (zu)^{i-1/2} f(u) du \int_0^\infty t K_{\nu}(it) K_{\nu}(ut) dt$$

$$= \frac{1}{\pi} \frac{(i)^{-\nu}}{\sin \nu \pi} \int_0^\infty (zu)^{i-\nu} f(u) (i2\nu u - u^{2\nu} - \frac{u^{2\nu}}{2^2}) du$$

$$= -K_{\nu, i} K_{\nu} [f'].$$  

The change of order of integration can be justified by absolute convergence. Thus, we obtain our first identity

$$K_{\nu, i} K_{\nu} + K_{\nu, i} K_{\nu} = 0$$

(3.2)

The identity given by (3.2) can alternatively be established by making use of the Mellin transform theory. That is, the Mellin transform of the iterated operator $K_{\nu, i} K_{\nu} [f]$, is given formally by

$$m(K_{\nu, i} K_{\nu} [f]) = m(-\frac{1}{\pi} x^{1/2} K_{\nu} (ix); s) m(-\frac{1}{\pi} x^{1/2} K_{\nu} (x); 1-s) f^*(s)$$

$$= \frac{(-i)^{s-\alpha/2}}{4\sin \frac{1}{2} \pi(s - \nu + \frac{1}{2}) \sin \frac{1}{2} \pi(s + \nu + \frac{1}{2})} f^*(s),$$

where $f^*(s)$ denotes the Mellin transform of $f(x)$. Also,

$$m(K_{\nu, i} K_{\nu} [f']) = m(-\frac{1}{\pi} x^{1/2} K_{\nu} (x); s) m(-\frac{1}{\pi} x^{1/2} K_{\nu} (-ix); 1-s) f^*(s)$$

$$= \frac{(-i)^{s-3/2}}{4\sin \frac{1}{2} \pi(s - \nu + \frac{1}{2}) \sin \frac{1}{2} \pi(s + \nu + \frac{1}{2})}.$$
Then
\[ m((K_{\nu,i} + K_{\nu,-i})[f]) = 0, \]

implying that
\[ K_{\nu,i}K_{\nu} + K_{\nu,-i}K_{\nu} = 0, \]
as shown above. Similarly, one can show that
\[ K_{\nu}K_{\nu,i} + K_{\nu}K_{\nu,-i} = 0, \]
a sort of conjugate of the identity in (3.2). Consider the representation
\[ K_{\nu}(ix) = \frac{\pi}{2\sin \nu\pi} \left( e^{-\frac{i\nu\pi}{2}} - e^{\frac{i\nu\pi}{2}} \right) \]
then
\[ K_{\nu,i}[f] = \int_0^\infty (xt)^{1/2} K_{\nu}(ix) f(t) dt, \quad |\nu| < 1 \]
\[ = \frac{\pi}{2\sin \nu\pi} \left( e^{-\frac{i\nu\pi}{2}} \int_0^\infty (xt)^{1/2} f(t) dt - e^{\frac{i\nu\pi}{2}} \int_0^\infty (xt)^{1/2} f(t) dt \right) \]
\[ = \frac{\pi}{2\sin \nu\pi} \left( e^{-\frac{i\nu\pi}{2}} H_{-\nu}[f] - e^{\frac{i\nu\pi}{2}} H_\nu[f] \right), \]
where \( H_\nu \) denotes the Hankel transform operator of order \( \nu \). Thus, in operator form,
\[ K_{\nu,i} = \frac{\pi}{2\sin \nu\pi} e^{-\frac{i\nu\pi}{2}} - \frac{\pi}{2\sin \nu\pi} e^{\frac{i\nu\pi}{2}} H_\nu \]
and similarly
\[ K_{\nu,-i} = \frac{\pi}{2\sin \nu\pi} e^{\frac{i\nu\pi}{2}} - \frac{\pi}{2\sin \nu\pi} e^{-\frac{i\nu\pi}{2}} H_\nu. \]
Now, after substituting for \( K_{\nu,i} \) and \( K_{\nu,-i} \) in (3.2) and rearranging, we have
\[ e^{-\frac{i\nu\pi}{2}} (H_{-\nu}K_{\nu} - K_{\nu}H_\nu) = e^{\frac{i\nu\pi}{2}} (H_\nu K_{\nu} - K_{\nu}H_{-\nu}). \]
By comparing the real and imaginary parts and solving, we obtain the identities
\[ K_{\nu}H_{-\nu} = H_{-\nu}K_{\nu} \]
and
\[ H_{\nu}K_{\nu} = K_{\nu}H_{-\nu} \]
Now,
\[ H_{1/2}[f] = \int_0^\infty (xt)^{1/2} \sqrt{2\pi} \sin(xt)f(t)\,dt \]
\[ = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \sin(xt)f(t)\,dt \]
\[ = S[f], \]
and similarly,
\[ H_{-1/2}[f] = C[f], \]
where \( S \) and \( C \) are the usual Fourier sine and cosine transform respectfully.

Also,
\[ K_{1/2}[f] = \int_0^\infty (xt)^{1/2} K_{1/2}(xt)f(t)\,dt \]
\[ = \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty e^{-xt}f(t)\,dt \]
\[ = \left(\frac{\pi}{2}\right)^{1/2} L[f], \]
where \( L[f] \) is the Laplace transform.

Hence, setting \( \nu = \frac{1}{2} \) in (3.4), we obtain the relations,
\[ LS = CL \]
and
\[ LC = SL. \]  \hspace{1cm} (3.7)

Incidently, a general relation involving the operators \( K_{\nu} \) and \( H_{\mu} \) can be established:
\[ H_{-\nu} K_{\nu} = \cosec \nu \pi \left( \frac{1}{2}(\nu - \mu) \pi K_{\mu} H_{\nu} + \sin \frac{1}{2}(\nu + \mu) \pi K_{\mu} H_{-\nu} \right) \]  \hspace{1cm} (3.8)

On setting \( \mu = \nu \), (3.8) yields the identities (3.5) and (3.6). Next, from the representation (3.4), we have symbolically
\[ K_{\nu, i}^2 = \left(\frac{\pi}{2\sin \nu \pi}\right)^2 \left( e^{-i\nu \pi} H_{-\nu} - e^{i\nu \pi} H_{\nu} \right)^2 \]
\[ = \left(\frac{\pi}{2\sin \nu \pi}\right)^2 (2\cos \nu \pi I - H_{-\nu} H_{\nu} - H_{\nu} H_{-\nu}) \]  \hspace{1cm} (3.9)

since \( H_{-\nu}^2 = H_{\nu}^2 = I \), the identity operator, with \( H_{\nu} \) being the Hankel transform.

Similarly, one can show that
\[ K_{\nu, i}^2 = K_{\nu, -i}^2. \]  \hspace{1cm} (3.10)
And, in the same vein, we have

\[ K_{\nu, i}K_{\nu, -i} + K_{\nu, -i}K_{\nu, i} = 2\left(\frac{\pi}{2\sin \nu \pi}\right)^2 \left\{ 2i - \cos \nu \pi (H_{\nu, -\nu} + H_{-\nu, \nu}) \right\} \]  
\( (3.11) \)

Now, going back to equation (3.1) and using the results given by (3.2), (3.3), and (3.9) in (3.11) and simplifying, we have finally, for \(|\nu| < 1, \)

\[ H_{\nu, -\nu} + H_{-\nu, \nu} = \left(\frac{2}{\pi}\sin \nu \pi\right)^2 K_{\nu}^2 = 2\cos \nu \pi I. \]  
\( (3.12) \)

An interesting relationship can be established by putting \( \nu = \pm \frac{1}{2} \) in (3.12). Then

\[ H_{1/2, -1/2} + H_{-1/2, 1/2} = \frac{4}{\pi^2} K_{1/2}^2 = 0 \]

or

\[ SC + CS = \frac{2}{\pi} L^2, \]  
\( (3.13) \)

where \( S, C, \) and \( L \) denote the Fourier sine, Fourier cosine and Laplace transforms respectively.

From (3.9) and (3.12), one can establish the identity

\[ K_{\nu}^2 = -K_{\nu, -i}^2 = -K_{\nu, i}^2, \]  
\( (3.14) \)

which, on setting \( \nu = \pm \frac{1}{2} \), yields (3.13).

REFERENCES

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