THE KRULL RADICAL, k-PRIMITIVE RINGS, AND CRITICAL RINGS

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Abstract. We generalize results on the Krull radical, k-primitive rings, and critical rings from rings with identity to rings which do not necessarily contain identity.

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1. Introduction. In this paper we extend some results on Krull dimension from rings with identity to rings which do not necessarily contain identity. The basic idea is to embed a ring \( R \) into the usual ring \( R_1 \) with identity, and to study the relation between the right ideals of \( R \) and of \( R_1 \).

In the first section of this paper we use Krull dimension to define the Krull radical of \( R \), denoted \( K(R) \). The Krull radical is a generalization of the Jacobson radical, and was first defined by Deshpande and Feller [1] for rings with identity. Our main result in this section is that \( K(R) = K(R_1) \). This enables us to use previous work in [1] to characterize the Krull radical as the annihilator of all critical \( R \)-modules, which in turn lets us determine the Krull radical of the \( n \times n \) matrix ring over \( R \). We then describe the relation between the Krull radical of \( R \) and that of a two-sided ideal \( I \subseteq R \). Finally we derive containment relations between the Krull radical on the one hand and the Jacobson and prime radicals on the other.

In the next section we look at k-primitive rings, which are a generalization of
primitive rings. We list the main properties of these rings and generalize slightly a theorem on these rings (Prop. 3.4). We finally turn our attention to critical rings, which are closely related to \( k \)-primitive rings. Necessary and sufficient conditions are given for a critical ring to be a domain, and those critical rings which are not domains are completely characterized.

In what follows the letter \( R \) denotes an associative ring which does not necessarily contain identity. An \( R \)-module \( M_R \) is a right \( R \)-module; usually we will simply call this module \( M \).

Let \( \mathbb{Z} \) denote the integers. We define \( R_1 = \{(r, n) \mid r \in \mathbb{R}, n \in \mathbb{Z}\} \), where addition is componentwise and multiplication is given by \((r, n) \cdot (r', n') = (rr' + nr' + n'r, nn')\). This is just the usual ring with identity in which \( R \) is embedded. For notational simplicity, we identify \( R \) with the subring \((R, 0)\) of \( R_1 \) to which \( R \) is isomorphic. All modules over \( R_1 \) are unital, so that every \( R \)-module \( M \) can be considered an \( R_1 \)-module if we define \( m(r, z) = mr + mz \) for all \( m \in M \), \((r, z) \in R_1 \). Conversely, any \( R_1 \)-module \( M \) can be considered an \( R \) module with scalar multiplication defined by \( mr = m(r, 0) \) for all \( m \in M \), \( r \in R \). Krull dimension for an \( R \)-module \( M \) is defined as in [2] and is denoted \( K \text{ dim } M \), or sometimes \( K \text{ dim } M_R \). A familiarity with the results of this paper is assumed. Note that for any \( R \)-module \( M \), \( K \text{ dim } M_R = K \text{ dim } M_{R_1} \), and \( M \) is a \( k \)-critical \( R \)-module if and only if \( M \) is a \( k \)-critical \( R_1 \)-module. Finally, \( E(M) \) denotes the injective hull of this module \( M \).

2. THE KRULL RADICAL.

As in [1] we say that a right ideal \( H \) of a ring \( R \) is \( k \)-co-critical if \( \frac{R}{H} \) is a \( k \)-critical \( R \)-module. A right ideal \( H \) of \( R \) is \( n \)-modular if there exist \( e \in R \), \( 0 \neq n \in \mathbb{Z} \), such that \( er - nr \in H \) for all \( r \in R \). If \( n = 1 \), then we call \( H \) modular in accordance with the usual terminology. A right ideal which is either maximal modular, \( 1 \)-co-critical and \( n \)-modular, or \( k \)-co-critical, \( k \geq 2 \), is called a special co-critical right ideal of \( R \). The Krull radical of \( R \), denoted \( K(R) \), is defined to be the intersection of all the special co-critical right ideal of \( R \), if any exist; if there are none, then we define \( K(R) = R \). Note that this definition of the Krull radical coincides with that given in [1] if \( R \) has identity. In order to be able to use the
results of [1], we first prove that $K(R) = K(R_1)$.

**Lemma 2.1** Let $H$ be a right ideal of $R$, $H \not\subseteq R$, and let $H_1 = \{(e, -n) \in R_1 \mid er - nr \in H \text{ for all } r \in R\}$. Then

1. $H_1$ is the unique right ideal of $R_1$ which is maximal with respect to the property that $H_1 \cap R = H$;
2. $H$ is the $n$-modular if and only if $H_1 \not\subseteq R$.

**Proof** (1) It is routine to verify that $H_1$ is a right ideal of $R_1$. The uniqueness of $H_1$ follows from the observation that $H_1 = \{x \in R_1 \mid x R \subseteq H\}$.

(2) This follows because $H_1 \not\subseteq R$ if and only if $H_1$ contains some $(e, -n) \in R_1$ with $n \not\in 0$.

**Lemma 2.2** Let $M$ be a trivial $R$-module; i.e., $mr = 0$ for every $m \in M$, $r \in R$. If $K \dim M = k$ exists, then $k \leq 1$.

**Proof** Since $M$ is a trivial $R$-module, its $R$-module structure is the same as its structure as an abelian group - i.e., as a $Z$-module. By [2, Cor. 4.4] $K \dim M \leq K \dim Z = 1$.

**Theorem 2.3** $K(R) = K(R_1)$.

**Proof** By [3, p. 11, Thm. 2] and by definition of the Krull radical, $K(R) \subseteq J(R_1) = J(R) \subseteq R$. Thus, $K(R_1) = \bigcap_{H_1 \in C} (H_1 \cap R)$, where $C$ is the set of co-critical right ideals of $R_1$. We will show that the set of special co-critical right ideals of $R$ coincides with the set of right ideals of the form $H_1 \cap R$, where $H_1 \in C$. Since $J(R) = J(R_1)$, we do not need to consider the case where $H_1$ is a maximal right ideal of $R_1$.

Suppose that $R$ has some special co-critical right ideals. Let $H$ be a special $k$-co-critical right ideal of $R$, $k > 0$, and let $H_1$ be as in Lemma 2.1. We first determine $K \dim \frac{R_1}{H_1}$. By [2, Lemma 1.1]
Now $\frac{R_1}{R + H_1}$ is a homomorphic image of $\frac{R}{R}$, which is a trivial $R$-module. Thus,

$$K \dim \frac{R_1}{R + H_1} = \sup \left[ K \dim \frac{R_1}{R + H_1}, K \dim \frac{R + H_1}{H_1} \right]$$

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$$K \dim \frac{R_1}{R + H_1} \leq K \dim \frac{R}{R} = 1$$

By lemma 2.2. Since $\frac{R + H_1}{H_1} \cong \frac{R}{R \cap H_1} = \frac{R}{H}$ we have

$$K \dim \frac{R_1}{H_1} = k.$$  

To show $H_1$ is co-critical, let $K_1$ be any right ideal of $R_1$ which properly contains $H_1$. Then $R \cap K_1$ properly contains $H$ because of the way $H_1$ is defined. Repeating the argument we used to find $K \dim \frac{R_1}{H_1}$ gives us that $K \dim \frac{R_1}{K_1} = K \dim \frac{R}{R \cap K_1} < K \dim \frac{R}{H} = k$. Therefore $H_1$ is a $k$-co-critical right ideal of $R_1$.

Conversely, suppose that $H_1$ is a $k$-co-critical right ideal of $R_1$, $k > 0$, and assume $H_1 \nsubseteq R$ (if there is no such $H_1$, then $R$ has no special co-critical right ideals contrary to our assumption). Let $H = R \cap H_1$. Since $\frac{R}{H} \cong \frac{R + H_1}{H_1} \subseteq \frac{R_1}{H_1}$ we have that $\frac{R}{H}$ is $k$-critical by [2, Prop. 2.3]. If $k \geq 2$ then $H$ is a special co-critical right ideal of $R$. Suppose $k = 1$. Then $H_1 \nsubseteq R$; for, if $H_1 \subseteq R$, then there is an onto map from $\frac{R_1}{H_1}$ to $\frac{R}{R}$, since both modules have Krull dimension 1, and $\frac{R_1}{H_1}$ is critical, we must have $\frac{R_1}{H_1} \cong \frac{R_1}{R}$. But then $\frac{R_1}{H_1} \cdot R = 0$, which implies $R \subseteq H_1$ and this in turn implies that $R \subseteq H_1 \cap R = H$, contradicting the fact that $K \dim \frac{R}{H} = 1$. Thus, $H_1 \nsubseteq R$. We must have, then, that $H_1$ contains some $(e, -n) \in R$ with $n > 0$, so for every $r \in R$, $(e, -n)r = er - nr \in H_1 \cap R = H$. Hence $H_1$ is $n$-modular, and therefore special.

Suppose now that $R$ has no special co-critical right ideals. Then $K(R) = R$ by definition. Since $\frac{R_1}{R}$ is 1-critical, $R$ is a co-critical right ideal of $R_1$. Every other co-critical right ideal $H_1$ of $R_1$ contains $R$; for, if $R \nsubseteq H_1$ then $H_1 \cap R$ is a special co-critical right ideal of $R$, contradiction. Therefore, $K(R_1) = R = K(R)$. 

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This completes the proof.

**COROLLARY 2.4**  
(1) $K(R)$ is the set of elements of $R$ which annihilate every critical right $R$-module.  
(2) $K(R)$ is a two sided ideal of $R$.  
(3) $K(R) = 0$.

**PROOF** This follows from Thm. 2.3 and [1, Thm. 2.1].

The next result shows that $K(R_n) = (K(R))_n$, where $R_n$ is the ring of $n \times n$ matrices over $R$. If $R$ has identity, then $E_{ij}$ denotes the matrix with 1 in the $(i, j)$ position and zeroes elsewhere.

**LEMMA 2.5.** Let $R$ be a ring with identity, and let $H$ be a right ideal of $R$. Take $H(1)$ to be the set of all matrices in $R_n$ whose $i$th row has entries from $H$ and whose other entries are arbitrary. Then $\frac{R_n}{H(1)}$ is a critical $R_n$-module if and only if $\frac{R}{H}$ is a critical $R$-module.

**PROOF** For simplicity, assume that $i = 1$. Note that $\frac{R_n}{H(1)}$ consists of matrices whose only non-zero row is the first. Thus, any submodule $S$ of $\frac{R_n}{H(1)}$ can be written $S = \begin{bmatrix} N_1 & \cdots & N_n \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$ where $N_1, \ldots, N_n$ are subsets of $R$. Now $N_1 = \ldots = N_n$; for,

$$\frac{R_n}{H(1)} E_{ij} \subseteq \frac{R_n}{H(1)}$$

for any $1 \leq j \leq n$, and $\frac{R_n}{H(1)} E_{jj}$ consists of matrices with nonzero entries in the $(1, j)$ position - i.e., from $N_j$ - and zeroes elsewhere. But then for any $1 \leq k \leq n$, $\frac{R_n}{H(1)} E_{jj} E_{jk} \subseteq \frac{R_n}{H(1)}$. This implies that $N_j \subseteq N_k$. Since $j$ and $k$ are arbitrary, we have that $N_1 = \ldots = N_n$. Call this set $N$. It is routine to check that $N$ is an $R$-submodule of $\frac{R}{H}$. Thus, there is a 1-1 onto order preserving map $f$ from the $R_n$-submodules of $\frac{R_n}{H(1)}$ to the $R$-submodules of $\frac{R}{H}$, given by $f: \begin{bmatrix} N & \cdots & N \\ 0 & \cdots & 0 \end{bmatrix} \rightarrow N$. The result follows immediately from this.

**LEMMA 2.6.** Let $R$ be a ring with identity. If $M$ is a cyclic critical $R_n$-module, then there is a co-critical right ideal $H(1) \subseteq R_n$ as in Lemma 2.5 such that $M$ is
isomorphic to \( \frac{R_n}{H(1)} \).

**PROOF** Since \( M \) is cyclic, we can find a matrix \( A \in M \) such that \( M = AR_n \). We show first that there is an integer \( j, 1 \leq j \leq n \), such that every element in \( M \) has non-zero \( j^{th} \) row. Suppose this is not the case. Then there is a collection of elements \( X_1, X_2, \ldots, X_n \in R_n \) such that \( AX_k \neq 0 \) and the \( k^{th} \) row of \( AX_k \) is zero. But then
\[
(AX_1)_n \cap \ldots \cap (AX_n)_n = 0,
\]
contradicting the fact that \( M \) is a uniform module by [2, Cor. 2.5 and 2.6].

We can assume without loss of generality that every non-zero element of \( M \) has non-zero first row. Let \( M' \) be the module consisting of all matrices whose first row appears as the first row of a matrix in \( M \), and whose other entries are zero. Define a map \( f: M \to M' \) as follows: If \( A \in M \), then \( f(A) \) is the matrix whose first row is the same as that of \( A \), and whose other entries are zero. This map is certainly an \( R_n \)-isomorphism and \( M' \) is of the appropriate form. This completes the proof.

**THEOREM 2.7.** \( K(R_n) = (K(R))^n \)

**PROOF** First assume \( R \) has identity. Let \( M \) be any cyclic critical \( R_n \)-module. By Lemma 2.6, \( M \cong \frac{R_n}{H(1)} \), where \( H(1) \) is defined as in Lemma 2.5. By Cor. 2.4 (2), \( \gamma(R) \) is a two-sided ideal of \( R \). Let \( X \in K(R_n) \), \( x \) the \((i,j)\) entry of \( X \). Then \( x = \frac{R}{H(1)} \times E_{ij} = E_{ij} \in K(R_n) \) so that \( \frac{R_n}{H(1)} \times E_{ij} = 0 \). As in the proof of Lemma 2.5, this shows that \( x \) annihilates the critical \( R \)-module \( \frac{R}{H} \). Since \( M \) is an arbitrary cyclic \( R_n \) module, so is \( \frac{R}{H} \); thus, by Cor. 2.4 (1), \( x \in K(R) \). Therefore, \( K(R_n) \subseteq (K(R))^n \). The reverse inclusion follows by reversing the steps of the argument. Hence \( K(R_n) = (K(R))^n \) when \( R \) has identity.

If \( R \) does not contain identity, embed \( R \) into \( R_1 \). From the previous paragraph and Thm. 2.2 we have \( (K(R))^n = (K(R_1))^n = K((R_1)_n) \). However, just as a critical module over \( R \) can be considered a critical module over \( R_1 \) and vice versa, so we can identify modules over \( R_n \) and \( (R_1)_n \). Therefore, by Cor. 2.4 (1), \( K((R_1)_n) = K(R_n) \). This completes the proof.

We now describe the relation between the Krull radical of a ring \( R \) and that of a
two-sided ideal $I$ in $R$.

**Lemma 2.8.** Let $R$ be a ring such that $R = K(R)$, let $I$ be a two-sided ideal of $R$, and let $M$ be a $k$-critical $I$-module. Then either $MI = 0$ or $MI$ is a $k$-critical $R$-module.

**Proof.** Assume $MI \not= 0$, and take $C$ to be a critical $R$-submodule of $MI$. Then $CR = 0$ by Cor. 2.4 (1), so $CI = 0$. Hence $K \dim R_C = K \dim C_I = k$, which implies that $K \dim MI_R \geq k$. Since the reverse inclusion always holds, we have $K \dim MI_R = k$. That $MI$ is a critical $R$-module follows from the fact that $MI$ is a critical $I$-module.

**Proposition 2.9.** Let $R$ be a ring such that $R = K(R)$, and let $I$ be a two-sided ideal of $R$. Then $K(1) \subseteq I$.

**Proof.** Let $M$ be a critical right $1$-module. If $MI \not= 0$, then there is some $i \in I$ for which $Mi \not= 0$. Since $MiR \subseteq MIR = 0$ by Lemma 2.8 and Cor. 2.4 (1), $Mi$ is a critical $R$-module. Hence the map $f: M \to Mi$ defined by $f(m) = mi$ for all $m \in M$ is actually an $I$-isomorphism. But then for any $m \in M$, $f(mi) = f(m)i = mi^2 = 0$ and hence $Mi = 0$, contradiction. Therefore $MI = 0$, and by Cor. 2.4 (1), $I = K(I)$.

**Example 2.10.** Prop. 2.9 is true if we substitute the Jacobson radical for the Krull radical. By [4, Thm. 48], this is equivalent to the fact that $J(I) = I \cap J(R)$ for any ideal $I$ of a ring $R$. Unfortunately, this does not hold for the Krull radical.

Let $R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}$ where $F$ is any field, $x$ is a commuting indeterminate over $F$, and the ring operations are the usual matrix addition and multiplication. By [1, Ex. 4], $K(R) = 0$. However, if we take $I = \begin{bmatrix} 0 & F[x] \\ 0 & 0 \end{bmatrix}$, then $K(I) = I$ because $I$ has no special co-critical right ideals; for, since $I_I$ is isomorphic to a direct sum of copies of $F$, any special co-critical right ideal would have to be maximal modular. Certainly $I$ has no such right ideal.

We now describe the containment relations between $K(R)$ on the one hand and $J(R)$ and $P(R)$ (the prime radical of $R$) on the other.
PROPOSITION 2.11. (1) For any ring $R$, $K(R) \subseteq J(R)$.

(2) If $R$ is a ring with Krull dimension, then $K(R) \subseteq P(R)$.

(3) If $R$ is a commutative ring, then $P(R) \subseteq K(R)$.

PROOF If we embed $R$ into $R_1$, then $P(R) = P(R_1)$, $J(R) = J(R_1)$, and $K(R) = K(R_1)$ by [4, Cor. after Thm. 59], [3, p. 11 Thm. 2], and Thm. 2.3 of this paper respectively. Hence we may assume that $R$ has identity. Now (1) follows from the definitions of $K(R)$ and $J(R)$, while (2) and (3) are mentioned in [1, p. 188] for rings with identity.

EXAMPLE 2.12. (1) The containments in Prop. 2.11 (1) and 2.11 (2) are both proper. Let $R$ be as in Ex. 2.10. Then $K(R) = 0$, but $P(R) \neq 0$.

(2) The containment in Prop. 2.11 (3) also is proper. Let $S = \mathbb{Z}_2[x_1, x_2, \ldots, x_n, \ldots]$ where $\{x_1, x_2, \ldots, x_n, \ldots\}$ is a countably infinite set of commuting indeterminates. Take $I$ to be the ideal generated by the polynomials $x_{2j-1}x_{2j} + x_{2j+1}x_{2j+2}$, $j = 1, 2, \ldots$ and let $R = S[I]$. Say that $x_{2j-1}x_{2j} = x$ in $R$ for all $j$. Then $x \notin P(R)$, but $x \in K(R)$ by [5, Ex. 4.17].

(3) In general, $P(R)$ and $K(R)$ are incomparable. Let

$$R = \begin{bmatrix} \mathbb{Z}_2 & S \\ I & I \end{bmatrix}, \quad a = \begin{bmatrix} 0 & \bar{x}_1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$$

where these symbols are defined in the previous paragraph. Then $a \in P(R)$, but $a \notin K(R)$; for, if $I'$ is the ideal of $S$ generated by all $\bar{x}_j$, $j > 1$, then $C = \begin{bmatrix} \mathbb{Z}_2 & S \\ (I' + I) & I \end{bmatrix}$ is critical but $C a \neq 0$. Now $b \notin P(R)$, but $b \in K(R)$, because a map $f: S[I] \to M$, where $M$ has Krull dimension, has kernel containing almost all the $\bar{x}_j$'s, and hence $x$.

3. CO-PRIMITIVE IDEALS

Just as $J(R)$ can be expressed as the intersection of certain two-sided ideals of $R$, so can $K(R)$. Let $H_1, H_2, \ldots, H_n$ be a finite collection of special co-critical right ideals of $R_1$, and suppose that $E(R) \approx E(R)$ for all $1 \leq j, k \leq n$. If $K \dim \frac{R}{H_j} = k$ for all $1 \leq j \leq n$, then the largest two-sided ideal $D \subseteq \bigcap_{j=1}^{n} H_j$ is called a $k$-co-primitive ideal of $R$. An ideal which is $k$-co-primitive for some ordinal $k$ is
called co-primitive. It is not hard to see that \( D = \{ r \in R \mid \frac{R_1}{(H_j)_1} r = 0 \text{ for all } 1 \leq j \leq n \} \). Here \((H_j)_1\) is the extension of \( H_j \) to a co-critical right ideal of \( R_1 \) as in Lemma 2.1.

**Theorem 3.1.** \( K(R) \) is the intersection of all the co-primitive right ideals of \( R \).

**Proof** From Cor. 2.4 (2), \( K(R) \) is a two-sided ideal of \( R \). Since \( K(R) \subseteq H \) for every special co-critical right ideal \( H \subseteq R \), then \( K(R) \subseteq D \) for every co-primitive ideal of \( R \). Thus, \( K(R) \subseteq \cap D \). Conversely, if \( r \in \cap D \), then by the observation previous to this theorem we have \( M \cdot r = 0 \) for any critical \( R \)-module \( M \). Thus, \( \cap D = K(R) \) by Cor. 2.4 (1), so \( K(R) = \cap D \).

If \( 0 \) is a \( k \)-co-primitive ideal of a ring \( R \) with Krull dimension \( k \), then \( R \) is said to be \( k \)-primitive. This definition coincides with that given in [6].

**Proposition 3.2.** Let \( R \) be a ring with Krull dimension \( k \). Then \( R \) is \( k \)-primitive if and only if \( R \) has a faithful critical finitely generated module \( C \) with \( K \dim R = K \dim C \).

**Proof** Suppose that \( R \) is \( k \)-primitive. Then there is a finite collection of special \( k \)-co-critical right ideals \( H_1, \ldots, H_n \) whose intersection is \( 0 \) and such that \( E(R_{H_j}^k) = E(R_{H_k}^j) \) for all \( 1 \leq j, k \leq n \). But then \( E(\frac{R_1}{(H_j)_1}) = E(\frac{R_1}{(H_k)_1}) \) so we may assume that each \( \frac{R_1}{(H_j)_1} \) lies in the same injective hull. The module \( C = \frac{R_1}{(H_1)_1} + \ldots + \frac{R_1}{(H_n)_1} \) is critical by [6, Lemma 3.1], finitely generated, and faithful, and \( K \dim R = k = K \dim C \). The converse follows by reversing the steps of this argument.

The main properties of \( k \)-primitive rings have been investigated in [6]. We list some of these properties here. Recall that the **assassinator** of a uniform module \( C \) over a ring \( R \) with Krull dimension is that ideal \( P \) which is maximal among the annihilators of submodules of \( C \).

**Theorem 3.3.** Let \( R \) be a \( k \)-primitive ring with faithful critical module \( C \), and
let P be the assassinator of C.

(1) If A, B = 0 for two right ideals A and B, then either A = 0 or B ⊆ P (i.e., R is P-primary);

(2) P is the only prime ideal of R which is not a large right ideal;

(3) if H is any non-zero right ideal of R, then K dim H = K dim R;

(4) R and C are nonsingular;

(5) if H is a large right ideal of R, then K dim \( \frac{R}{H} \) < K dim R;

(6) the injective hull of R is a simple artinian ring.

In [7, Thm. 3.4], Boyle, Deshpande and Feller characterize a k-primitive piecewise domain (PWD) which contains a faithful critical right ideal. (We shall refer to this type of ring as a BDF ring after the authors.) This result can be used to describe a slightly broader class of rings. Recall that a PWD R is a ring with identity which contains a complete set of orthogonal idempotents \( e_1, \ldots, e_n \) such that if \( x \in e_i R e_j, y \in e_j R e_k \), then \( x y = 0 \) implies \( x = 0 \) or \( y = 0 \). In what follows, we assume that R is written as an \( n \times n \) upper triangular matrix ring; see [8]. Recall also that a ring S is a quotient ring of R if R is a large R-submodule of S.

In the next result, we assume that R is a noetherian k-primitive ring with identity which is a direct sum of non-isomorphic critical right ideals (and hence is a PWD by [8]). Since \( E(R) \) is a matrix ring over a division ring D with identity 1, we can define the matrix \( M = E_{11} + \ldots + E_{nn} \) where \( E_{ij} \) is the matrix with 1 in the \((i,j)\) position and 0's elsewhere, \( 1 \leq j \leq n \).

**PROPOSITION 3.4** Let R and M be as above. Then R has a quotient ring \( S = R + RMR \) which is a noetherian BDF ring if and only if \((RMR) \subseteq RMR + R \) and \( RMR \) is a finitely generated \( R \)-module.

**PROOF** Note that R is an upper triangular matrix ring with \( e_j R e_k \neq 0 \) for \( j < k \). Also, each \( e_j R e_j \) is noetherian, for if \( I = \sum_{k>j} e_k R + \sum_{j>k} e_j R e_k \), then \( e_j R e_j \cong \frac{R}{I} \).

Finally, note that \( RMR = e_1 S \).

Let \( S = R + RMR \). Assume that \( RMR \) is a finitely generated \( R \)-module and that \((RMR) \subseteq R + RMR \). Since \( S \subseteq E(R) \), S is a quotient ring of R. Also, S is a finitely generated \( R \)-module.
generated $R$-module, which implies that $S$ is a noetherian ring and that $e_1 S$ is a 
finite generated $R$-module. Now $e_1 S \subseteq E(e_1 R)$ which is uniform, so $e_1 S$ is a critical 
$R$-module by [9, Cor. 2.4]. Let $0 \neq H$ be an $S$-submodule of $e_1 S$. Then

$$K \dim \left( \frac{e_1 S}{H} \right)_S \leq K \dim \left( \frac{e_1 S}{H} \right)_R \leq K \dim (e_1 S) . \tag{3.1}$$

But $K \dim (e_1 S) = K \dim (e_1 S)_R$; for, since $R$ is a PWD, $e_1 S_n$ is merely a sum of copies 
of $e_1 R e_n$. Further,

$$(e_1 S_n)_S = (e_1 S_n)_R e_n = (e_1 S_n)_R \tag{3.2}$$

Thus, $K \dim (e_1 S) = K \dim (e_1 S_n)_R \leq K \dim (e_1 S)$. So that $K \dim (e_1 S) = K \dim (e_1 S)_R$.

This together with (3.1) shows that $e_1 S$ is a critical $S$-module. Now $e_1 S$ is faithful; 
for, if $e_1 S s = 0$ for some $s \in S$, then for any idempotents $e_j, e_k \in R$ we have

$e_1 S e_j e_k = 0$. Since $S$ is a PWD, $e_j e_k = 0$. Therefore, $s = 0$.

Conversely, let $S = R + RMR$ be a noetherian BDF ring. Since $S_S$ is noetherian 
$(e_1 S_n)_S$ is noetherian. But by (3.2), $(e_1 S_n)_R$ is noetherian. Let

$$S' = e_1 S_{n-1} + e_1 S_n e_1 S_{n-1} \tag{3.3}$$

because $S' = S_{n-1} e_1 S_{n-1}$, so that $(e_1 S_{n-1} + e_1 S_n)$ is noetherian. Continuing 
in this manner, we have $e_1 S_R$ noetherian, and hence $RMR_R$ is finitely generated.

Finally, since $S$ is a ring, $(RMR)^2 \subseteq R + RMR$.

Prop. 3.4 applies more generally to a ring $R$ with identity which is a direct sum 
of non-singular non-isomorphic critical right ideals; such a ring is a direct sum of 
ideals, each of which is a $k$-primitive ring by [10, Prop. 5.3].

**Example 3.5.** (1) Let $F$ be a field, $x$ a commuting indeterminate over $F$, and let

$$R = \begin{bmatrix} F & 0 & F[x] \\ 0 & F & F[x] \\ 0 & 0 & F[x] \end{bmatrix}$$

with the usual matrix operations. Then $R$ satisfies the
conditions of Prop. 3.4. If
\[
\begin{bmatrix}
F & F & F[x] \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
then \( S = R + RMR = \begin{bmatrix}
F & F & F[x] \\
0 & F & F[x] \\
0 & 0 & F[x]
\end{bmatrix} \) is a BDF ring.

(2) Let \( x \) and \( y \) be commuting indeterminates over \( F \), and let
\[
R = \begin{bmatrix}
F[x] & 0 & F[x, y] \\
0 & F[y] & F[x, y] \\
0 & 0 & F[x, y]
\end{bmatrix}
\]
and
\[
RMR = \begin{bmatrix}
F[x] & F[x, y] & F[x, y] \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Then \( S = \begin{bmatrix}
F[x] & F[x, y] & F[x, y] \\
0 & F[y] & F[x, y] \\
0 & 0 & F[x, y]
\end{bmatrix} \) has no Krull dimension, since \( F[x, y] \) does not have finite uniform dimension as an \( F[y] \) module. In this case \( RMR \) is not finitely generated.

4. CRITICAL RINGS

A ring \( R \) is critical if \( R_R \) is a critical \( R \)-module. If \( R \) has identity, then \( R_R \) is faithful, and hence \( R \) is \( k \)-primitive. Thus, we could describe the structure of this ring using Thm. 3.3. However, it is possible to prove more about \( R \), even if we do not assume that \( R \) has identity.

PROPOSITION 4.1. If \( R \) is a domain with Krull dimension, then \( R \) is critical.

PROOF Let \( C \) be a critical right ideal of \( R \), \( 0 \neq c \in C \). The map \( f: R \to C \) given by \( f(r) = cr \) is 1-1, proving \( R \) is critical.

The converse of Prop. 4.1 is true if \( R \) has identity. To examine this converse for \( k \)-critical rings which do not possess identity, we need to consider separately the cases \( k > 1 \) and \( k = 1 \). Recall that a module \( M \) is monoform if, for any submodule \( N \subseteq M \), a homomorphism \( f: N \to M \) is either zero or 1-1. Any critical module is monoform by [2, Cor. 2.5].

PROPOSITION 4.2. If \( R \) is a \( k \)-critical ring, \( k > 1 \), then \( R \) is a domain.
PROOF Let $T = \{ r \in R \mid rR = 0 \}$. By Lemma 2.2, $K \dim T = 1 < K \dim R$, contradiction. Hence $T = 0$. Now if $a, b \in R$ with $ab = 0$, then either $b = 0$ or $a \in T$ because $R$ is monoform, so $a = 0$.

To examine 1-critical rings, we need the following notation: $Q$ is the set of rational numbers, $G(p) = \{ \frac{a}{k} \mid a \in Q, p \text{ a fixed prime} \}$, and $Z_p = \frac{G(p)}{Z}$.

**THEOREM 4.3.** Let $R$ be a 1-critical ring. Then the following are equivalent:

1. $R$ is a domain;
2. $R^2 + 0$;
3. $0$ is an $n$-modular right ideal of $R$.

**PROOF**

1. $\Rightarrow$ (2) Trivial.

2. $\Rightarrow$ (3) Assume $R^2 + 0$. If there is $0 + n \in Z$, $0 + r \in R$ such that $nr = 0$, then $nr = 0$ for all $r \in R$ because $R$ is monoform and so $0$ is $n$-modular. Otherwise, since $R^2 + 0$, we can pick $x \in R$ such that $xr + 0$ for any $0 + r \in R$. Now the module $xR + xZ = xR \cap xZ$, being a proper homomorphic image of the 1-critical module $xR + xZ$, is artinian. Hence $xZ \cap xR + 0$. In particular, there exist $e \in R$, $n \in Z$ such that $0 + x e = nx$. Multiply on the right by any $r \in R$ and cancel the element $x$ to show that $0$ is $n$-modular.

3. $\Rightarrow$ (1) In this part of the proof we use the argument from [11, Prop. 4.1]. Let $0$ be an $n$-modular ideal of $R$; i.e., there are $e \in R$, $0 + n \in Z$ such that $er - nr = 0$ for all $r \in R$. Let $T = \{ r \in R \mid rR = 0 \}$. As in Prop. 4.2, we show that $T = 0$. If $nr = 0$ for some $0 + r \in R$, then $nR = 0$; in particular, any element $t \in T$ generates a finite, hence artinian, right ideal of $R$. Hence $t = 0$, so that $T = 0$. Now suppose that $nr + 0$ for any $0 + r \in R$. We note first that $\frac{R}{T}$ is a domain; for let $a, b \in R$ with $ab \in T$. If $b \notin T$, then $bR + 0$. Since $abR = 0$, the fact that $R$ is monoform implies that $aR = 0$ and so $a \in T$. Hence $\frac{R}{T}$ is a domain. Because $R$ is 1-critical, $\frac{R}{T}$ is an artinian domain, and hence is a division ring $D$.

Define a group homomorphism $f: \frac{R}{T} \rightarrow T$ by $f(r + T) = rt$ for all $r \in R$, where $0 + t \in T$ is fixed but arbitrary. This map is 1-1; for if $rt = 0$ for some $r \in R$, $r \notin T$, then $R = 0$ because $R$ is monoform, and hence $r \in T$, contradiction. Hence $T$
contains a subgroup isomorphic to D. Now R, and hence D, has no elements of finite order by assumption. This implies that D, and hence T, has a subgroup which is isomorphic to Q. Without loss of generality we write Q ⊆ T. Thus, Q is a trivial R-module, which implies that \( \text{K dim } Q_R = \text{K dim } Q_Z \). However, K dim \( Q_Z \) does not exist, contradiction. It follows that T = 0, and R is a domain. This completes the proof.

The case when 0 is a maximal modular right ideal is handled similarly. Hence we may summarize:

**COROLLARY 4.4.** Let R be a critical ring. Then R is a domain if and only if 0 is a special co-critical right ideal of R; otherwise, \( R^2 = 0 \).

**THEOREM 4.5.** Let R be a l-critical ring satisfying \( R^2 = 0 \). Then as a group R is isomorphic to a finite sum of \( Z \) and \( G(p)'s \) for various primes p.

**PROOF** Since the injective hull of R is isomorphic to Q, identify R with some subgroup of Q. If G is finitely generated, then G is isomorphic to \( Z \) by [13, Thm. 9.24]. If G is not finitely generated, then \( \frac{G + Z}{Z} \) is isomorphic to a direct sum of \( Z_p 's \) and \( Z_{n'}s \), where there is a distinct summand for every prime p which divides b for some \( \frac{a}{b} \in G \).

**EXAMPLE 4.6.** Let \( R = \{ \frac{a}{2^k} | a, k \in Z \} \) where the product of any two elements is 0. Then R is l-critical but not right noetherian.

We note that Hein [12] has recently generalized Thm. 4.5.

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