ON SEPARABLE ABELIAN EXTENSIONS OF RINGS

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ABSTRACT. Let R be a ring with 1, G (= \langle p_1 \rangle \times ... \times \langle p_m \rangle) a finite abelian automorphism group of R of order n where \langle p_1 \rangle is cyclic of order n_1 for some integers n, n_1, and m, and C the center of R whose automorphism group induced by G is isomorphic with G. Then an abelian extension R[x_1, ..., x_m] is defined as a generalization of cyclic extensions of rings, and R[x_1, ..., x_m] is an Azumaya algebra over K (= C^G = \{ c in C / (c)p_1 = c for each p_1 in G\}) such that R[x_1, ..., x_m] \cong R^G \otimes_K C[x_1, ..., x_m] if and only if C is Galois over K with Galois group G (the Kanzaki hypothesis).

KEY WORDS AND PHRASES. Abelian ring extensions, separable algebras, Azumaya algebras, Galois extensions.

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1. INTRODUCTION.

Cyclic extensions of rings have been intensively investigated by Nagahara and Kishimoto [1], Parimula and Sridharan [2], the present author [3,4,5], and others. In [3], a separable cyclic extension R[x] with respect to a cyclic automorphism group \langle p \rangle of R of order n for some integer n over a noncommutative ring R was studied. It was shown ([3], Theorem 3.3) that if R is Galois over R^\langle p \rangle (= \{ r in R / (r)p = r \}) with Galois group \langle p \rangle and if R^\langle p \rangle is contained in the center C of R, then R[x] is an Azumaya algebra over R^\langle p \rangle, where x^n (= b for some b in R) and n are units in R^\langle p \rangle. Let G be an abelian automorphism group of R of order n such that G = \langle p_1 \rangle \times ... \times \langle p_m \rangle.
where \( \langle \rho_1 \rangle \) is a cyclic subgroup of order \( n_1 \) for some integers \( n, m, \) and \( n_1 \).

Noting that \( (C)\rho_1 = C \) for each \( \rho_1 \), we shall study an abelian extension

\[ R[x_1, \ldots, x_m] \]

with respect to \( G \), where \( rx_i = x_i(r\rho_1) \) for each \( r \in R \), \( x_i^n = b_i \) which is a unit in \( C^G \), \( x_i x_j = x_j x_i \) for all \( i \) and \( j \), and the set \( \{x_1^{k_1} \cdots x_m^{k_m} \mid 0 \leq k_1, \ldots, k_m < n_1 \} \) is a basis over \( R \). A ring \( R \) is called to satisfy the Kanzaki hypothesis ([6], P. 110) if \( R \) is Azumaya over \( C \) with a finite automorphism group \( G \) and \( C \) is Galois over \( K (= C^G) \) with Galois group induced by and isomorphic with \( G \). DeMeyer [7] has shown that \( R \cong R^G \otimes_K C \) under the Kanzaki hypothesis for \( R \). The present paper will generalize the Parimula-Sridharan theorem from cyclic extensions ([2], Proposition 1.1, [3], Theorem 3.3) to abelian extensions \( R[x_1, \ldots, x_m] \) with respect to an abelian automorphism group \( G (= \langle \rho_1 \rangle \times \cdots \times \langle \rho_m \rangle) \) of \( R \). Let \( G \) restricted to \( C \) be isomorphic with \( G \). Then we shall show that \( C \) is Galois over \( K (= C^G) \) if and only if \( R[x_1, \ldots, x_m] \) is an Azumaya algebra over \( K \) such that \( R[x_1, \ldots, x_m] \cong R^G \otimes_K C[x_1, \ldots, x_m] \) where \( R^G \) is an Azumaya \( K \)-algebra. Thus, a structure of \( R[x_1, \ldots, x_m] \) is obtained. Moreover, a structure of \( C[x_1, \ldots, x_m] \) is also obtained when each direct summand of \( G \) is a \( G \)-subgroup (see definition below).

2. PRELIMINARIES.

Throughout, let \( R \) be a ring with \( 1 \), \( C \) the center of \( R \), \( G (= \langle \rho_1 \rangle \times \cdots \times \langle \rho_m \rangle) \) an abelian automorphism group of \( R \) of order \( n \) where \( \rho_1 \) is cyclic of order \( n_1 \) for some integers \( n, n_1, \) and \( m \). Then \( R[x_1, \ldots, x_m] \) is the abelian extension of \( R \) with respect to \( G \) as defined in Section 1. We denote \( C^G \) by \( K \), and assume that the automorphism group of \( C \) is isomorphic with \( G \). The Azumaya algebra \( R \) is called to satisfy the Kanzaki hypothesis ([6], P. 110) if \( C \) is Galois over \( K \) with Galois group induced by and isomorphic with \( G \).

For separable extensions, Azumaya algebras, and Galois extensions, see [3], [4], and [5].

3. ABELIAN EXTENSIONS.

Keeping the notations of Sections 1 and 2, we shall show the Parimula-Sridharan theorem ([2], Proposition 1.1, [3], Theorem 3.3) and two structural theorems for abelian extensions \( R[x_1, \ldots, x_m] \). We begin with a proposition on separable abelian extensions.
PROPOSITION 3.1. Let $G = (\langle \sigma_i \rangle_{i=1}^{m})$ be an abelian automorphism group of $R$ of order $n$. If $n$ and $x_i = b_i$ are units in $C^G$ for each $i$, then $R[x_1, \ldots, x_m]$ is a separable extension of $R$.

PROOF. Since $n_i$ divides $n$, $n_i$ is a unit in $C^G$. Hence the cyclic extension $R[x_1]$ with respect to $\langle \sigma_1 \rangle$ is a separable extension over $R$ ([3], Lemma 3.1). Now $\langle \sigma_2 \rangle$ is extended to an automorphism group of $R[x_1]$ by $(x_1)^{\sigma_2} = x_1$, so $(R[x_1])[x_2]$ is a separable extension over $R[x_1]$ by a similar reason. Thus $R[x_1, x_2] = (R[x_1])[x_2]$ is a separable extension over $R$ by the transitivity of separable extensions. By repeating the above argument $(m-2)$ times, $R[x_1, \ldots, x_m]$ is a separable extension over $R$.

We now show the Parimala-Sridharan theorem for $R[x_1, \ldots, x_m]$.

THEOREM 3.2. By keeping the notations of Proposition 3.1, if $R$ satisfies the Kanzaki hypothesis, then $R[x_1, \ldots, x_m]$ is an Azumaya $K$-algebra.

PROOF. By Proposition 3.1, $R[x_1, \ldots, x_m]$ is a separable extension over $R$. By the Kanzaki hypothesis for $R$, $R$ is separable over $C$ and $C$ is Galois over $K$, so $R[x_1, \ldots, x_m]$ is a separable extension over $K$ by the transitivity of separable extensions. So, it suffices to show that the center of $R[x_1, \ldots, x_m]$ is $K$. It is easy to see that $K$ is contained in the center.

Since $\{x_1^{k_1} \cdots x_m^{k_m} / 0 \leq k_i \leq n_i \}$ is a basis of $R[x_1, \ldots, x_m]$ over $R$, we can take $f$ in the center of $R[x_1, \ldots, x_m]$ such that $f = a_o x_1^{k_1} \cdots x_m^{k_m}$ where $a_o$ and $a$ are in $R$, and $0 \leq k_i \leq n_i$. Then, $rf = fr$ for each $r$ in $R$. This implies that $ra_o = a_o r$ and $ar = (r)f_1^{k_1} \cdots f_m^{k_m}$. Hence $a_o$ is in $C$, and the second equation implies that $a(r-(r)f_1^{k_1} \cdots f_m^{k_m}) = 0$ for each $r$ in $C$. Thus $a$ is in the annihilator ideal $I$ of $\{r-(r)f_1^{k_1} \cdots f_m^{k_m} / r \in C \}$ of $R$. Since $R$ is Azumaya over $C$, $I = I_0 R$ where $I_0$ is the annihilator ideal of $\{r-(r)f_1^{k_1} \cdots f_m^{k_m} / r \in C \}$ of $C$. $I_0 = \{0\}$ ([7], Proposition 1.2) because $C$ is Galois over $K$ with Galois group induced by and isomorphic with $G$. Thus $I = \{0\}$, and so $a = 0$. Therefore, $f = a_o$ in $C$. Also, $x_i f = f x_i$ for each $i$, so $a_o = (a_o)f_i^{k_i}$ for each $i$. Thus $a_o$ is in $K$. This completes the proof.

Next is a structural theorem for $R[x_1, \ldots, x_m]$ under the Kanzaki hypothesis.
THEOREM 3.3. If $R$ satisfies the Kanzaki hypothesis, then $R[x_1, \ldots, x_m] \cong R\otimes_K C[x_1, \ldots, x_m]$ as Azumaya $K$-algebras.

PROOF. By Proposition 3.1, $C[x_1, \ldots, x_m]$ is an Azumaya algebra over $K$. Then, similar to the arguments used in the proof of Theorem 3.2, we shall show that the commutant of $C[x_1, \ldots, x_m]$ in $R[x_1, \ldots, x_m]$ is $R^G$. Clearly, $R^G$ is contained in the commutant. Now, let $f = a_0 + x_1 \ldots x_m a$ be an element in the commutant for some $a_0$ and $a$ in $R$ and $0 \leq k_i < n_i$. Then $cf = fc$ for each $c$ in $C$. This implies that $a = 0$. Also, $x_i f = f x_i$ for each $i$, so $a_0$ is in $R^G$.

Thus $f (= a_0)$ is in $R^G$. Noting that $C[x_1, \ldots, x_m]$ and $R[x_1, \ldots, x_m]$ are Azumaya algebras over $K$, we have that $R[x_1, \ldots, x_m] \cong R\otimes_K C[x_1, \ldots, x_m]$ by the well known commutant theorem for Azumaya algebras ([7], Theorem 4.3, P. 57).

COROLLARY 3.4. If $R$ satisfies the Kanzaki hypothesis, then $R^G$ is an Azumaya algebra over $K$.

PROOF. This is a consequence of Theorem 3.3 and the commutant theorem for Azumaya algebras.

We are going to show a converse of Theorem 3.3.

THEOREM 3.5. If $R[x_1, \ldots, x_m]$ is an Azumaya algebra over $K$ such that $R[x_1, \ldots, x_m] \cong R\otimes_K C[x_1, \ldots, x_m]$ where $R^G$ is an Azumaya $K$-algebra, then $C$ is Galois over $K$ with Galois group induced and isomorphic with $G$.

PROOF. By the commutant theorem for Azumaya algebras, since $R[x_1, \ldots, x_m]$ and $R^G$ are Azumaya $K$-algebras, so is $C[x_1, \ldots, x_m]$. Then, we claim that $C$ is Galois over $K$ with Galois group $G$. Suppose not. There is a non-identity $g$ in $G$ such that $\{c-\text{(c)g} / c \in C\}$ is not $C$ ([7], Proposition 1.2). Let $g = g_1^{k_1} \ldots g_m^{k_m}$ for some $k_i, 0 \leq k_i < n_i$. Since $I$ generated by $(c-\text{(c)g})$ for $c$ in $C$ is a $G$-ideal of $C$ (that is, $(I)^G = I$), we have an Azumaya algebra $(C/I)[x_1, \ldots, x_m]$ over $K/(K \cap I)$. On the other hand, one can show that $(x_1^{k_1} \ldots x_m^{k_m})$ is in the center of $(C/I)[x_1, \ldots, x_m]$. This is a contradiction. Thus $C$ is Galois over $K$ with Galois group $G$.

Let $S$ be a ring Galois extension over a subring $T$ with a finite Galois group $G$. A normal subgroup $H$ of $G$ is called a $G$-subgroup if $S$ is Galois over $S^H$ with Galois group $H$ and $S^H$ is Galois over $T$ with Galois group $G/H$. Keep-
ing the notations of Theorem 3.5, we give a structural theorem for $C[x_1, \ldots, x_m]$. We denote the center of $C[x_1, \ldots, x_i-1, x_i+1, \ldots, x_m]$ by $C_i$ for each $i$.

Clearly, $C_i = C$. Let each direct summand of $G$ be a $G$-subgroup, we have:

**THEOREM 3.6.** If $C$ is Galois over $K$ with Galois group $G$, then the abelian extension $C[x_1, \ldots, x_m] \cong C[x_1] \otimes_K \cdots \otimes_K C_m[x_m]$ as Azumaya $K$-algebras.

**PROOF.** Extending $\rho_i$ from $C$ to $C[x_1, \ldots, x_m]$ by $(x_j)\rho_i = x_j$ for each $i$ and $j$, we claim that $C[x_1, \ldots, x_m] \cong (C[x_1, \ldots, x_{m-1}]^{\rho_m})_K C_m[x_m]$. In fact, since $C$ is Galois over $K$, $C$ is Galois over $K$ with Galois group $(G/\langle \rho_m \rangle)$ is a $G$-subgroup of $G$ by hypothesis. Now, the center of $C[x_1, \ldots, x_{m-1}]$ is $C$ so $C[x_1, \ldots, x_{m-1}]$ satisfies the Kanzaki hypothesis; that is, $C[x_1, \ldots, x_{m-1}]$ has an automorphism group $\langle \rho_m \rangle$ such that its center $C$ is Galois over $(C = K)$ with Galois group $\langle \rho_m \rangle$ induced by and isomorphic with $\langle \rho_m \rangle$. But $C[x_1, \ldots, x_m] \cong (C[x_1, \ldots, x_{m-1}]^{\rho_m})_K C_m[x_m]$, so $C[x_1, \ldots, x_m] \cong (C[x_1, \ldots, x_{m-1}]^{\rho_m})_K C_m[x_m]$ by Theorem 3.3. Next, considering $(C[x_1, \ldots, x_{m-1}]^{\rho_m})$, we have that $(C[x_1, \ldots, x_{m-1}]^{\rho_m}) \cong (C[x_1, \ldots, x_{m-1}]^{\rho_m})$ such that the center of $C[x_1, \ldots, x_{m-1}] = C_m$ which is Galois over $K$ with Galois group $\langle \rho_m \rangle$. Since $\langle \rho_m \rangle$ is an automorphism group of $C_m[x_1, \ldots, x_{m-2}]$, $C_m[x_1, \ldots, x_{m-2}]$ satisfies the Kanzaki hypothesis with a center which is Galois over $K$ with Galois group $\langle \rho_m \rangle$. Hence $C_m[x_1, \ldots, x_{m-2}] \cong C_m[x_1, \ldots, x_{m-2}]^{\rho_m} \otimes_K C_{m-1}[x_{m-1}]$. The above arguments can be repeated for $(m-2)$ more times. Thus the proof is completed.

As immediate consequences of Theorem 3.5 and Theorem 3.6, we have the following:

**COROLLARY 3.7.** If $R$ satisfies the Kanzaki hypothesis such that each direct summand of $G$ is a $G$-subgroup, then $R[x_1, \ldots, x_m] \cong R \otimes_K C_1[x_1] \otimes_K \cdots \otimes_K C_m[x_m]$.

**COROLLARY 3.8.** If $R$ satisfies the Kanzaki hypothesis such that the center $C$ of $R$ has no idempotents but $0$ and $1$, then $R[x_1, \ldots, x_m] \cong R \otimes_K C_1[x_1] \otimes_K \cdots \otimes_K C_m[x_m]$.

**PROOF.** Since $C$ is Galois over $K$ with no idempotents but $0$ and $1$, each direct summand of $G$ is indeed a $G$-subgroup ([6], Theorem 1.1, P. 80, or [8]).
REFERENCES


