RESEARCH NOTES

ALMOST CONVEX METRICS AND
PEANO COMPACTIFICATIONS

R.F. DICKMAN, JR.

Department of Mathematics
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia 24061 U.S.A.

(Received June 23, 1981)

ABSTRACT. Let \((X,d)\) denote a locally connected, connected separable metric space. We say the \(X\) is \(S\)-metrizable provided there is a topologically equivalent metric \(\rho\) on \(X\) such that \((X,\rho)\) has Property S, i.e., for any \(\epsilon > 0\), \(X\) is the union of finitely many connected sets of \(\rho\)-diameter less than \(\epsilon\). It is well-known that \(S\)-metrizable spaces are locally connected and that if \(\rho\) is a Property S metric for \(X\), then the usual metric completion \((\tilde{X},\tilde{\rho})\) of \((X,\rho)\) is a compact, locally connected, connected metric space; i.e., \((\tilde{X},\tilde{\rho})\) is a Peano compactification of \((X,\rho)\). In an earlier paper, the author conjectured that if a space \((X,d)\) has a Peano compactification, then it must be \(S\)-metrizable. In this paper, that conjecture is shown to be false; however, the connected spaces which have Peano compactifications are shown to be exactly those having a totally bounded, almost convex metric. Several related results are given.

KEY WORDS AND PHRASES. Almost Convex Metrics, Property S metrics, Peano spaces, Compactifications.

1980 MATHEMATICS SUBJECT CLASSIFICATIONS CODES. 54F25, 54E35.

1. INTRODUCTION.

Throughout this note let \((X,d)\) denote a metric space. We say that \(d\) is convex...
provided that, for any pair \(x, y \in X\), there is \(z \in X\) such that \(d(x, z) = d(z, y) = d(x, y)/2\).

It is almost convex if, for \(x, y \in X\) and \(\varepsilon > 0\), there is \(z \in X\) such that \(\left|d(x, z) - d(x, y)/2\right| < \varepsilon\) and \(\left|d(z, y) - d(x, y)/2\right| < \varepsilon\) \([1, 2]\).

We say that \(X\) is S-metrizable provided there is a topologically equivalent metric \(\rho\) on \(X\) such that \((X, \rho)\) has Property S, i.e., for any \(\varepsilon > 0\), \(X\) is the union of finitely many connected sets of \(\rho\)-diameter less than \(\varepsilon\). It is well-known that S-metrizable spaces are locally connected and that if \(\rho\) is a Property S metric for \(X\), then the usual metric completion \((\tilde{X}, \tilde{\rho})\) of \((X, \rho)\) is a compact, locally connected, connected metric space, i.e., \((\tilde{X}, \tilde{\rho})\) is a Peano compactification of \((X, \rho)\) \([3, p. 154]\).

It is a famous result of R. H. Bing that any continuous curve \(P\) (i.e., a compact, locally connected, connected metric space) can be assigned a convex metric \([1]\).

In an earlier paper \([4]\), the author conjectured that, if \(X\) is locally connected and if \(X\) has a Peano compactification, then \(X\) is S-metrizable. In this paper we show, by example, that this conjecture is false; however, we do obtain a characterization of such spaces in terms of the existence of a totally bounded, almost convex metric for \(X\). We also obtain several related results characterizing totally bounded (S-metrizable, almost convex) metrics.

2. PEANO COMPACTIFICATIONS.

THEOREM 2.1. A connected metric space \((X, d)\) has a Peano compactification if and only if it has a topologically equivalent totally bounded, almost convex metric.

PROOF. The necessity. Let \((P, \tilde{r})\) be a Peano compactification of \(X\), i.e., \(P\) is a continuous curve and \(X\) is a dense subset of \(P\). By R. H. Bing's result, there exists an equivalent metric \(\tilde{\rho}\) for \(P\) such that \(\tilde{\rho}\) is convex. It then follows that \(\sigma = \rho\) | \(X\) is totally bounded and almost convex; cf. \([1, Thm. 10]\).

The Sufficiency. Let \(\tilde{r}\) be an almost convex, totally bounded metric for \(X\). Let \((\tilde{X}, \tilde{r})\) be the usual metric completion of \((X, r)\). We will argue that \((\tilde{X}, \tilde{r})\) is a Peano compactification of \((X, r)\). Clearly, \(\tilde{X}\) is compact since \(\tilde{r}\) is totally bounded.

Furthermore, \(\tilde{r}\) is a convex metric for \(\tilde{X}\); let \(x, y \in \tilde{X}\). Since \(\tilde{r}\) is almost convex, there exists a sequences \(x_1, x_2, \ldots, y_1, y_2, \ldots\), and \(z_1, z_2, \ldots\) in \(X\) such that

\[|\tilde{r}(x_n, z_n) - \tilde{r}(x_n, y_n)/2| < 2^{-n}\] and \[|\tilde{r}(z_n, y_n) - \tilde{r}(x_n, y_n)/2| < 2^{-n}\]
Since $r$ is totally bounded, without loss of generality, we may assume that each of the sequences $x_1, x_2, \ldots, y_1, y_2, \ldots,$ and $z_1, z_2, \ldots$ is Cauchy in $X.$ Then by the completeness of $(X, r),$ it follows that $\lim_{n \to \infty} x = x$ and $\lim_{n \to \infty} y = y.$ Furthermore, if $\lim_{n \to \infty} z = z,$ $r(x, z) = r(z, y) = r(x, y)/2$ since $r$ is continuous. Thus $r$ is convex and complete. It follows from Theorem 3.1 of [5] that the spheres $S_{r}(x, \varepsilon)$ of $X$ are connected sets. This implies that $X$ is locally connected and this completes the proof.

**EXAMPLE 2.1.** Let $P$ be the square $\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ in the plane. For $n \in \mathbb{N},$ let $L_n = \{(1/n, y) : 0 \leq y \leq 1\}$ and let $L_0 = \{(0, y) : 0 \leq y \leq 1\}.$ Set $X = P \setminus \bigcup_{n=1}^{\infty} L_n.$ Then $P$ is a Peano compactification of $X$; however, $X$ is not $S$-metrizable.

Suppose $\rho$ is an $S$-metric for $X$ and let $A = \{(x, 1) : 0 \leq x \leq 1\}$ and $B = \{(x, 0) : 0 \leq x \leq 1\}.$ Then $A$ and $B$ are compact and hence $\rho(A, B) = \varepsilon > 0.$ Now the components $C_1, C_2, \ldots$ of $X \setminus (A \cup B)$ have limit points in each of $A$ and $B.$ Thus, any collection of connected sets of $\rho$-diameter less than $\varepsilon/3$ that covers a component $C_n$ has at least one such connected subset lying entirely in $C_n.$ This implies that $\rho$ is not an $S$-metric for $X$; however, if $d$ is the relative metric on $X$ inherited from the usual metric on $P,$ $d$ is almost convex and totally bounded.

3. **RELATED RESULTS.**

A **compatible normal sequence** in a space $Z$ is a sequence $U_1, U_2, \ldots$ of open covers of $Z$ such that $U_{n+1}$ star-refines $U_n$ for $n = 1, 2, \ldots$ and so, for any $x \in Z,$ \{$St(x, U_n) : n = 1, 2, \ldots$\} is a neighborhood base for $x$ [5].

**THEOREM 3.1.** [6, Prop. 23.4] A $T_0$-space is metrizable if and only if it has a compatible normal sequence.

**COROLLARY 3.1.** A metric space $X$ is totally bounded if and only if $X$ has a compatible normal sequence $U_1, U_2, \ldots$ where each $U_n$ is a finite cover of $X.$

**PROOF.** Suppose $(X, d)$ is totally bounded. It follows from the total boundedness of $(X, d)$ that there is a finite open cover $U_1$ of $X$ such that $\delta_d(U) = 1/3$ for all $U \subseteq U_1$ where $\delta_d(U) = \sup \{d(x, y) : x, y \in U\},$ the $d$-diameter of $U.$ Since $U_1$ is finite, there is a Lebesgue number $\varepsilon_1 < 3^{-2}$ such that, if $d(x, y) < \varepsilon_1,$ then $x$ and $y$ be in some member of $U_1.$ Again, by the total boundedness of $(X, d),$ there is a finite open cover $V_1$ of $X$ such that $\delta_d(V) < \varepsilon_1.$ If $\varepsilon_2 < \varepsilon_1$ is a Lebesgue number for $V_1$ and $U_2$ is any fini
open cover of $X$ such that $\delta_d(U) < \varepsilon_2$ for any $U \in U_n$, then $U_2$ star-refines $U_1$. Continue in this manner and obtain a compatible normal sequence $U_1, U_2, \ldots$ for $X$.

On the other hand, suppose $U_1, U_2, \ldots$ is a compatible normal sequence for $X$ where each $U_n$ is finite. Then, in the usual metric $\rho$ for $X$ that is associated with $U_1, U_2, \ldots$ as given by S. Willard [6], $\delta_\rho(U) < 2^{n-1}$ and $U \in U_n$, $n = 2, 3, \ldots$. It then follows that, since each $U_n$ is finite, $\rho$ is a totally bounded metric for $X$. This completes the proof.

**COROLLARY 3.2.** A metric space $(X, d)$ is $S$-metrizable if and only if it has a compatible normal sequence $U_1, U_2, \ldots$ where each $U_n$ is a finite cover and the members of $U_n$ are connected sets.

**PROOF.** The necessity follows from the argument above, together with the observation that the covers $U_1, U_2, \ldots$ can be selected so as to consist of finitely many open and connected sets.

The sufficiency. We observe that, if $U_1, U_2, \ldots$ is a compatible normal sequence for $X$ where each $U_n$ is finite and the members of $U_n$ are connected sets and if $\rho$ is the usual metric associated with $U_1, U_2, \ldots$ as given in [6], then, for $U \in U_n$, $\delta_\rho(U) < 2^{n-1}$, $n = 2, 3, \ldots$ and the sets $U \in U_n$ are connected. Thus, for any $\varepsilon > 0$ and $k \in \mathbb{N}$ so that $0 < 2^{-k} < \varepsilon$, $x = \cup\{U : U \in U_k\}$ is a finite cover of $X$ by connected sets of $\rho$-diameter less than $\varepsilon$. This completes the proof.

**THEOREM 3.2** [2]. A connected metric space $X$ has an almost convex metric if and only if it has a compatible normal sequence $U_1, U_2, \ldots$ such that (i) each pair of points that is covered by either an element of $U_{n+1}$ or the union of a pair of intersecting elements of $U_{n+1}$ can be covered by an element of $U_n$ and (ii) each pair of points that can be covered by an element of $U_n$ can be covered by the union of two intersecting elements of $U_{n+1}$.

It is, apparently, very difficult to combine the total boundedness (finiteness) conditions of Corollaries 3.1 and 3.2 and the intersection-type properties of Theorem 3.2. It would be very desirable to do so in light of the results of the previous section.
REFERENCES


