TENSOR PRODUCTS OF COMMUTATIVE BANACH ALGEBRAS

U. B. TEWARI, M. DUTTA and SHOBHA MADAN

Department of Mathematics
Indian Institute of Technology
Kanpur-208016, U.P.
INDIA

ABSTRACT. Let $A_1$, $A_2$ be commutative semisimple Banach algebras and $A_1 \otimes A_2$ be their projective tensor product. We prove that, if $A_1 \otimes A_2$ is a group algebra (measure algebra) of a locally compact abelian group, then so are $A_1$ and $A_2$. As a consequence, we prove that, if $G$ is a locally compact abelian group and $A$ is a commutative semi-simple Banach algebra, then the Banach algebra $L^1(G,A)$ of $A$-valued Bochner integrable functions on $G$ is a group algebra if and only if $A$ is a group algebra. Furthermore, if $A$ has the Radon-Nikodym property, then the Banach algebra $M(G,A)$ of $A$-valued regular Borel measures of bounded variation on $G$ is a measure algebra only if $A$ is a measure algebra.

KEY WORDS AND PHRASES. Commutative semisimple Banach algebra, projective tensor product, group algebra, measure algebra, locally compact abelian group, Radon-Nikodym property.


1. INTRODUCTION.

Let $A$ be a commutative Banach algebra. We shall say that $A$ is a group algebra (measure algebra) if $A$ is isometrically isomorphic to $L^1(G)$ ($M(G)$) for some locally compact abelian group $G$. Let $G$ be a locally compact abelian group and $A$ be a commutative semi-simple Banach algebra. The space $L^1(G,A)$ of $A$-valued Bochner integrable functions on $G$ becomes a commutative Banach algebra (see [1], [2] and [3]). A natural question arises: when is $L^1(G,A)$ a group algebra? If $A = L^1(H)$ for some
locally compact abelian group $H$, then it is well known (Theorem 3.2 of [3])
that $L^1(G,A)$ is isometrically isomorphic to $L^1(G \times H)$. Thus $L^1(G,A)$ is a
group algebra if $A$ is a group algebra. We shall prove that the converse is also true.

There is another way of looking at this problem. It is known that $L^1(G,A)$ is iso-
metrically isomorphic to $L^1(G) \otimes A$ (see 6.5 of [4]). Thus, if $A_1$ and $A_2$ are group
algebras, then so is $A_1 \otimes A_2$. Conversely, we shall show that, if $A_1$ and $A_2$ are
two commutative Banach algebras and $A_1 \otimes A_2$ is a group algebra, then so are $A_1$
and $A_2$. It seems proper to remark that we are concluding properties for $A_1$ and $A_2$,
assuming corresponding properties for $A_1 \otimes A_2$. This is in contrast to the approach
of Gelbaum [5] and [6]. Our result for $L^1(G,A)$ readily follows from this. The
main tool in our investigation is a theorem of Rieffel [7] characterizing group
algebras. In this paper, Rieffel also characterized measure algebras. Accordingly,
we investigate whether the fact that $A_1 \otimes A_2$ is a measure algebra implies that $A_1$
and $A_2$ are measure algebras. We shall show that this is indeed the case. As a
consequence, we shall show that, if $A$ is a commutative Banach algebra having the
Radon Nikodym property and $M(G,A)$ is the Banach algebra of $A$-valued regular Borel
measures of bounded variation on $G$, then $M(G,A)$ is a measure algebra only if $A$ is
a measure algebra.

2. PRELIMINARIES.

Let $E$ and $F$ be Banach spaces. The projective tensor product of $E$ and $F$ (see
[8]) is denoted by $E \otimes F$. Every element $t \in E \otimes F$ can be expressed as
$t = \sum_{i=1}^{\infty} e_i \otimes f_i$, with each $e_i \in E$ and $f_i \in F$, such that $\sum_{i=1}^{\infty} \|e_i\| \|f_i\| < \infty$. The
norm of $t$ is given by

$$\|t\| = \inf \{ \sum_{i=1}^{\infty} \|e_i\| \|f_i\| : t = \sum_{i=1}^{\infty} e_i \otimes f_i \},$$

where the infimum is taken over all possible expressions of $t$.

Let $f \in E^*$, $g \in F^*$, and $t \in E \otimes F$ with $t = \sum_{i=1}^{\infty} e_i \otimes f_i$. We define

$$f \circ t = \sum_{i=1}^{\infty} f(e_i) f_i \text{ and } g \circ t = \sum_{i=1}^{\infty} g(f_i) e_i.$$  
The series defining $f \circ t$ and $g \circ t$ converge because $\sum_{i=1}^{\infty} \|e_i\| \|f_i\| < \infty$. It is obvious that $\|f \circ t\| \leq \|f\| \|t\|$ and
$$\|g \circ t\| \leq \|g\| \|t\|.$$  
The norms here, as well as elsewhere, refer to the norms
in the spaces containing the elements. \( t \rightarrow g \circ t \) and \( t \rightarrow f \circ t \) define bounded linear maps from \( E \otimes F \) to \( E \) and \( E \otimes F \) to \( F \), respectively. These maps will be frequently used in the sequel.

Let \((S, \Sigma, \lambda)\) be a measure space and \( X \) be a Banach space. \( L^1(S, X) \) denotes the Banach space of \( X\)-valued functions integrable with respect to \( \lambda \). We shall often use the fact that \( L^1(S) \otimes X \) is isometrically isomorphic to \( L^1(S, X) \).

Gelbaum [5] and Tomiyama [9] have shown that, if \( A \) and \( B \) are commutative Banach algebras, then \( A \otimes B \) forms a commutative Banach algebra whose maximal ideal space is homeomorphic to the cartesian product of the maximal ideal spaces of \( A \) and \( B \). The maximal ideal space of a commutative Banach algebra \( A \) will be denoted by \( \Delta(A) \). An element of \( \Delta(A) \) will be regarded as a multiplicative linear functional (m.l.f.) of \( A \). All the Banach algebras in our discussion will be taken to be commutative and semisimple. It is proved in [6] that \( A \otimes B \) has an identity if and only if both \( A \) and \( B \) have identities. It is also known [6] that \( A \otimes B \) is Tauberian if \( A \) and \( B \) are Tauberian. The following lemma, though simple, does not seem to have appeared in print.

**Lemma 2.1.** If \( A \otimes B \) is Tauberian, then so are \( A \) and \( B \).

**Proof.** Let us show that \( B \) is Tauberian. It can be shown in the same way that \( A \) is Tauberian. Let \( b \in B \) and \( \varepsilon > 0 \). Take \( \phi \in \Delta(A) \) and \( a \in A \) such that \( \phi(a) = 1 \). Let \( t = a \otimes b \). Choose \( s \in A \otimes B \) such that \( \hat{s} \) has compact support \( K \) and \( \|s-t\| < \varepsilon \).

Let \( K_1 = \{ \psi \in \Delta(B) : (\phi, \psi) \in K \} \). Then \( K_1 \) is compact. Let \( x = \phi \circ s \). Then \( \hat{x} \) is supported in \( K_1 \) and

\[
\|b-x\| = \|\phi \circ t - \phi \circ s\| \leq \|t-s\| < \varepsilon.
\]

This proves that \( B \) is Tauberian.

Let \((S, \Sigma)\) be a measurable space and \( X \) be a Banach space. Let \( \mu \) be an \( X\)-valued set function on \( \Sigma \). The total variation \( V(\mu) \) of \( \mu \) is defined for any \( E \subseteq S \) as follows.

\[
V(\mu)(E) = \sup \left\{ \sum_{1}^{n} \|\mu(E_i)\| : E_i \text{ disjoint}, E_i \subseteq E \text{ for } i \leq i \leq n \right\},
\]

the supremum being taken for all possible choices of \( E_i \)
An \( X \)-valued measure on \( (S, \mathcal{X}) \) is a countably additive set function from \( \mathcal{X} \) into \( X \). \( \mu \) is said to be of bounded variation if \( V(\mu) \) is finite. The space \( \overline{M}(S, \mathcal{X}, X) \) of \( X \)-valued measures of bounded variation on \( S \) forms a Banach space under the norm \( \| \mu \|_V = V(\mu)(S) \).

Let \( \lambda \) be a positive measure on \( (S, \mathcal{X}) \) and \( L^1(S, X) \) be the Banach space of \( X \)-valued functions on \( S \), integrable with respect to \( \lambda \). If \( F \in L^1(S, X) \), then we can define the mapping \( \mu_F: \mathcal{X} \rightarrow X \) by \( \mu_F(E) = \int_E F d\lambda \). Then \( \mu_F \) is an \( X \)-valued measure of bounded variation on \( S \). Let \( \mu \in \overline{M}(S, \mathcal{X}, X) \). We say that \( \mu \) has the derivative \( F \) with respect to \( \lambda \) if \( \mu \) equals \( \mu_F \) for \( F \in L^1(S, X) \). We say that \( X \) has the Radon-Nikodym property (\( X \) has RNP) if every \( X \)-valued measure \( \mu \) of bounded variation on an arbitrary measurable space \( (S, \mathcal{X}) \) has a derivative with respect to \( V(\mu) \). If \( X \) is separable and the dual of a Banach space or is reflexive, then \( X \) has RNP (see [10] and [11]). An example of a separable Banach space which does not have RNP is \( L^1[0,1] \) (see [12]).

Let \( G \) be a locally compact abelian group and let \( A \) be a commutative Banach algebra. \( M(G, A) \) denotes the Banach space of \( A \)-valued regular Borel measures of bounded variation on \( G \). Suppose the range of every \( \mu \in M(G, A) \) is separable. This is true if \( A \) has RNP or if \( G \) is second countable. Under these conditions, we can define the convolution of measures \( \mu \) and \( \nu \) belonging to \( M(G, A) \). This makes \( M(G, A) \) a commutative Banach algebra (see [13]). The algebra \( L^1(G, A) \) is an ideal in \( M(G, A) \) (see [14]). There is a natural isometric isomorphism from \( M(G) \otimes A \) into \( M(G, A) \) (Theorem 4.2 of [15]). This is a Banach algebra isomorphism and, if \( A \) has RNP, then it is onto (Theorem 4.4 of [15]).

Let \( \Delta(A) \) be a locally compact semisimple Banach algebra and \( m \in \Delta(A) \). Let \( P_m = \{ a \in A: \Re(a) = m(\|a\|) \} \). Then \( P_m \) is a cone in \( A \) and therefore introduces an order in \( A \). Let \( R_m = \{ a-b: a, b \in P_m \} \). \( m \) is said to be \( L^1 \)-inducing if the following conditions are satisfied:

1. \( \| m \| = 1 \).
2. \( P_m \) is a lattice.
3. If \( a, b \in R_m \) and \( a \wedge b = 0 \), then \( \| a+b \| = \| a-b \| \).
(4) If $a \in A$, then there exists unique elements, $a_1, a_2 \in \mathbb{R}_m$, such that
\[ a = a_1 + i a_2. \]

(5) Let $|a| = V |\text{Re} (e^{i\theta} a)|$: $\theta \in [0,2\pi]$. Then $||a|| = |||a|||$. 

[V and $\wedge$ respectively denote supremum and infimum. $\text{Re} (a) = a_1$ where $a = a_1 + i a_2$, $a_1 \in \mathbb{R}_m$. We note that if (1) - (3) hold, then $\mathbb{R}_m$ forms a real abstract $L$-space in the sense of Kakutani [16], and hence $\mathbb{R}_m$ is a boundedly complete lattice (see page 35 of [7]). Therefore, $|a|$ is well defined.

In [7], a $L'$-inducing m.l.f. is defined to be a m.l.f. which satisfies the following condition in addition to (1) - (5).

(6) For $a,b \in A$, $|a.b| \leq |a| \cdot |b|.$

However, White [17] has shown that a m.l.f. satisfying (1) - (5) automatically satisfies (6), and hence our definition is equivalent to that of [7]. We now state Rieffel's characterization of a group algebra.

**Theorem R.** Let $A$ be a commutative semisimple Banach algebra. $A$ is a group algebra if and only if

(a) every m.l.f. of $A$ is $L'$-inducing, and

(b) $A$ is Tauberian.

Let $A$ be a commutative semisimple Banach algebra and let $D$ be the collection of $L'$-inducing m.l.f.'s of $A$. Consider the $\omega^*$-topology on $D$. A continuous function $p$ on $D$ is said to be a D-Eberlein function if there exists a constant $k > 0$ such that for any choice of points $m_1, \ldots, m_n$ of $D$ and scalars $\alpha_1, \ldots, \alpha_n$; we have
\[ \sum_{1}^{n} \alpha_1 p(m_1) \leq k \|\sum_{1}^{n} \alpha_1 m_1\|_A^*. \]

The following theorem of Rieffel characterizes a measure algebra.

**Theorem R.** Let $A$ be a commutative Banach algebra and let $D$ be the set of $L'$-inducing m.l.f.'s of $A$. Then $A$ is a measure algebra if and only if

(i) $D$ is a separating family of linear functionals of $A$,

(ii) $D$ is locally compact in the $\omega^*$-topology, and

(iii) every D-Eberlein function is the restriction to $D$ of the Gelfand transform of some element of $A$. 
The 'if' part is nothing but Theorem B of [7]. The 'only if' part follows from the following and the familiar properties of Fourier-Stieltjes transforms.

**PROPOSITION 2.1.** The L'-inducing m.l.f.'s of M(G) are precisely those that are given by $\Gamma$, the dual of G.

**PROOF.** Let $S$ be the structure semigroup of M(G) (see 4.3 of [18]). M(G) can be identified (3.2 of [18]) with a weak*-dense subalgebra of M(S). Under this identification, the m.l.f.'s of M(G) are given by $\hat{S}$, the collection of semicharacters of S. Let $f \in \hat{S}$. Then, using the arguments of Proposition 2.5 of [7] (see also Proposition 2.8 of [7]), we can prove that f represents an L'-inducing m.l.f. if and only if $|f(s)| = 1$ for all $s \in S$. By 4.3.3 of [18], $|f|^{-1}$ is the canonical image of $\Gamma$ in S. This proves our proposition.

3. **MAIN RESULT.**

Our main result is the following theorem. All other results are derived as a consequence of this.

**THEOREM 3.1.** Let $A_1, A_2$ be commutative semisimple Banach algebras and $A = A_1 \otimes A_2$. Let $\varphi \in \hat{\mathcal{A}}$ be given by $(\cdot, \cdot)$ for $\psi \in \hat{\mathcal{A}_1}$ and $\tau \in \hat{\mathcal{A}_2}$. Then $\varphi$ is L'-inducing if and only if $\psi$ and $\tau$ are L'-inducing.

**PROOF.** Suppose $\varphi$ is L'-inducing. We shall show that $\psi$ satisfies (1) - (5) for $\psi$ to be L'-inducing. Since $1 = |\eta_\psi| \geq |\phi| = |\eta| \leq 1$, it follows that $|\eta_\psi| = |\eta| = 1$. Let $P = \{t \in A: \eta(t) = |t|^2\}$ and $P_\psi = \{r \in A_1: \phi(r) = |r|^2\}$. Choose a fixed $s \in A_2$ such that $\gamma(s) = |s|^2 = 1$. Let $t \in P_\psi$ and $r = \gamma(t) = |t|^2 = |\gamma(t)|$. Then $\phi(r) = \phi(t) = |t|^2 = |\gamma(t)|$. On the other hand, if $r \in P_\psi$, then $\phi(r \otimes s) = \phi(r) \gamma(s) = \phi(r) = |t|^2 = |r|^2 = |s|^2 = |r \otimes s|$, and so $r \otimes s \in P_\psi$. Thus we have shown that, if $t_1 \otimes t_2 \in P$ and $t_1 \geq t_2$, then $t_1 \vee t_2 = t_2$, and if $r_1, r_2 \in A_1$ and $r_1 \otimes s \geq r_2 \otimes s$, then $r_1 \otimes s \geq r_2 \otimes s$.

Now, let $r_1, r_2 \in P$. Then it is easy to see that $r_1 \vee r_2 = r_1 \vee r_2$ and $r_1 \wedge r_2 = r_1 \wedge r_2$. For example, if $r = r_1 \vee r_2$, then $r_1 \vee r_2 = r_1 \vee r_2$. Similarly, $r = r_2$. On the other hand, if $r' = r_1$ and $r' = r_2$, then $r' \otimes s = r_1 \otimes s$ and $r' \otimes s = r_2 \otimes s$. Therefore, $r' = r$. Note that $r_1 \vee r_2$ and
TENSOR PRODUCTS OF COMMUTATIVE BANACH ALGEBRAS

\( r_1 \wedge r_2 \) depend only on \( r_1 \) and \( r_2 \) and not on \( s \). Therefore, \( P_\phi \) is a lattice. We can also see that \((r_1 \vee r_2) \otimes s = (r_1 \otimes s) \vee (r_2 \otimes s) \) and \((r_1 \wedge r_2) \otimes s = (r_1 \otimes s) \wedge (r_2 \otimes s) \). For example, it is obvious that \((r_1 \vee r_2) \otimes s \geq (r_1 \otimes s) \vee (r_2 \otimes s) \) and furthermore,

\[
\|(r_1 \vee r_2) \otimes s - (r_1 \otimes s) \vee (r_2 \otimes s)\| = \eta \|((r_1 \otimes s) \wedge (r_2 \otimes s)) - (r_1 \vee r_2) \otimes s - (r_1 \otimes s) \vee (r_2 \otimes s)\| = 0.
\]

Next, if \( t \in R_\eta \) and \( r \in R_\phi \), then \( \psi \circ t \in R_\phi \) and \( r \otimes s \in R_\eta \). Moreover, all the above relations are true for \( r_1 \vee r_2 \) and \( r_1 \wedge r_2 \) for \( r_1, r_2 \in R_\phi \). Now, let \( r_1, r_2 \in R_\phi \) and \( r_1 \wedge r_2 = 0 \). Then \( r_1 \otimes s, r_2 \otimes s \in R_\eta \) and \((r_1 \otimes s) \wedge (r_2 \otimes s) = 0 \). Therefore,

\[
\|r_1 \otimes s + r_2 \otimes s\| = \|r_1 \otimes s - r_2 \otimes s\|, \text{ and hence } \|r_1 + r_2\| = \|r_1 - r_2\|.
\]

Hence \( \phi \) satisfies (3).

Suppose now that \( r \in A_1 \). Then \( r \otimes s \in A \) and \( r \otimes s = t_1 + it_2 \), with \( t_1, t_2 \in R_\eta \).

Then \( r = \psi \circ (r \otimes s) = \psi \circ t_1 + \psi \circ t_2 \). Also, if \( r = r_1 + ir_2 = r_3 + ir_4 \) for \( r_1 \in R_\phi \), then \( r \otimes s = r_1 \otimes s + i r_2 \otimes s = r_3 \otimes s + i r_4 \otimes s \). Therefore,

\[
r_1 \otimes s = r_3 \otimes s \text{ and } r_2 \otimes s = r_4 \otimes s.
\]

Hence, \( r_1 = r_3 \) and \( r_2 = r_4 \). We have also shown that \((\text{Re } r) \otimes s = \text{Re } (r \otimes s) \). Thus \( \psi \) satisfies (4). We now show that \( \phi \) satisfies (5). Let \( r \in A_1 \). First, we show that \( |r| = \psi \circ |r \otimes s| \) and \( |r \otimes s| = |r| \otimes s \).

We have

\[
\psi \circ |r \otimes s| - \text{Re } (e^{i\theta} r) = \psi \circ |r \otimes s| - \psi \circ (\text{Re } (e^{i\theta} r \otimes s)) = \psi \circ (|r \otimes s| - \text{Re } (e^{i\theta} r \otimes s)),
\]

for every \( \theta \in [0,2\pi) \). Therefore, \( \psi \circ |r \otimes s| \geq |r| \). On the other hand,

\[
|r| \geq \text{Re } (e^{i\theta} r). \text{ Hence } |r| \otimes s \geq \text{Re } (e^{i\theta} r) \otimes s = \text{Re}(e^{i\theta} (r \otimes s)). \text{ Therefore, } |r| \otimes s \geq |r \otimes s|, \text{ so that } |r| \geq \psi \circ |r \otimes s|.
\]

Thus we have \(|r| = \psi \circ |r \otimes s| \).

Also, since \( |r| \otimes s \geq |r \otimes s| \), we get

\[
\| |r| \otimes s - |r \otimes s| \| = \| |r| \otimes s - |r \otimes s| \| = 0.
\]

Therefore, \( |r| \otimes s = |r \otimes s| \). Now \(|r| = |r| \| |s| = |r \otimes s| \| = |r \otimes s| \| = |r| \| |s| \|. \text{ This proves that } \phi \text{ satisfies (5). Hence } \phi \text{ is L'-inducing.}

We can show similarly that \( \psi \) is L'-inducing.
Conversely, suppose \( \phi \) and \( \psi \) are \( L' \)-inducing. We shall show that \( \eta \) is \( L' \)-inducing. It is obvious that \( \| \eta \| = 1 \). Since \( \phi \) is \( L' \)-inducing m.l.f. of \( A_1 \), by Proposition 2.3 of [7], there exists a locally compact Hausdorff space \( X \) and a positive regular Borel measure \( \mu \) on \( X \) such that \( A_1 \) is isometrically linear isomorphic and, under the order induced by \( \phi \), order isomorphic to \( L^1(X, \mu) \). The dual of \( A_1 \) is then represented by \( L^\infty(X, \mu) \) and, under this representation, \( \phi \) is represented by the constant function \( \| \phi \| = 1 \) on \( X \). Now, \( A_1 \otimes A_2 = L^1(X, \mu) \otimes A_2 = L^1(X, \mu, A_2) \). Hereafter, we shall not distinguish between elements of \( A \) and \( L^1(X, \mu, A_2) \) and, for \( F \in L^1(X, \mu, A_2) \), statements like "\( F \in A_a \)" will be used without explanation. For \( F \in A \), we observe that \( F \in P \) if and only if \( \phi \circ F \in P_\psi \). This is so, because

\[
\| F \| \geq \| \phi \circ F \| \geq \| \psi (\phi \circ F) \|.
\]

We also have

\[
\| F \| = \int \| F(x) \| \, d\mu(x) \geq \psi \left( \int F(x) \, d\mu(x) \right) = \int \psi(F(x)) \, d\mu(x).
\]

This shows that \( F \in P_\eta \) if and only if \( F(x) \in P_\psi \) a.e. \((\mu)\). Let \( F_1, F_2 \in P_\eta \). Using the continuity and other properties of the lattice operations, it is easy to prove that the function \( F_1 \vee F_2 \) defined a.e. \((\mu)\) by \((F_1 \vee F_2)(x) = F_1(x) \vee F_2(x)\), belongs to \( L^1(X, \mu, A_2) \) and consequently defines an element of \( P_\eta \). This proves that \( P_\eta \) is a lattice. Other details involved in showing that \( \eta \) is \( L' \)-inducing are also now easy to verify and hence we omit them. This completes the proof of our Theorem.

Having proved our main theorem, we now proceed to give its consequences.

**Theorem 3.2.** Let \( A_1 \) and \( A_2 \) be commutative semisimple Banach algebras. Then \( A_1 \otimes_\eta A_2 \) is a group algebra if and only if \( A_1 \) and \( A_2 \) are group algebras.

**Proof.** As mentioned in the introduction, it is well known that, if \( A_1 \) and \( A_2 \) are group algebras, then so is \( A_1 \otimes_\eta A_2 \). The converse follows from Lemma 2.1, Theorem 2.1 and Theorem 3.1.

The following is an immediate consequence of Theorem 3.2.

**Theorem 3.3.** Let \( G \) be a locally compact abelian group and let \( A \) be a commutative semisimple Banach algebra. Then \( L^1(G, A) \) is a group algebra iff \( A \) is a group algebra.
PROOF. The result follows from Theorem 3.2 and the fact that the Banach algebras $L^1(G,A)$ and $L^1(C) \Theta \alpha A$ are isometrically isomorphic.

THEOREM 3.4. Let $A_1$ and $A_2$ be commutative semisimple Banach algebras and $A = A_1 \Theta \alpha A_2$. If $A$ is a measure algebra, then $A_1$ and $A_2$ are measure algebras.

PROOF. Let $D, D_1, D_2$ be the set of $L'$-inducing m.l.f.'s of $A, A_1,$ and $A_2$ respectively. Theorem 3.1 implies that $D = D_1 \times D_2$. Since $D$ satisfies condition (i) of Theorem 3.2, it easily follows that $D_1$ and $D_2$ also satisfy this condition. Since $D$ is locally compact in the $\omega^*$-topology, $D_1$ and $D_2$ are also locally compact in the $\omega^*$-topology. It remains to show that $A_1$ and $A_2$ satisfy condition (iii) of Theorem 3.2. We shall do this for $A_2$, the case of $A_1$ being similar. Since $A$ is a measure algebra, it has an identity. It follows that $A_1$ and $A_2$ have identities. Let $e$ be the identity of $A_1$. Let $p$ be a $D_2$-Eberlein function. Define the function $P$ on $D$ by $P(\phi, \psi) = p(\psi)$. Obviously, $P$ is continuous. Moreover,

$$\left| \sum \alpha_1 P(\phi_1, \psi_1) \right| = \left| \sum \alpha_1 p(\psi_1) \right| \leq k \left\| \sum \alpha_1 \psi_1 \right\|_{A_2^*}.$$

However, for any $a \in A_2$,

$$\langle a, \sum \alpha_1 \psi_1 \rangle = \langle e \otimes a, \sum \alpha_1 (\phi_1, \psi_1) \rangle \leq \left\| \sum \alpha_1 (\phi_1, \psi_1) \right\|_{A_1^*} \|a\|.$$

Therefore, $\left\| \sum \alpha_1 \psi_1 \right\|_{A_2^*} \leq \left\| \sum \alpha_1 (\phi_1, \psi_1) \right\|_{A_1^*}$. This shows that $P$ is a $D$-Eberlein function and therefore there exists $t \in A$ such that $\hat{e}(\eta) = P(\eta)$ for every $t \in \Delta(A)$.

Choose $\phi \in \Delta(A_1)$ and let $b = \phi \circ t$. Then $S(\psi) = p(\psi)$ for every $\psi \in \Delta(A_2)$. This shows that $A_2$ satisfies condition (iii) of Theorem 3.2 and the proof of our theorem is complete.

THEOREM 3.5. Let $G$ be a locally compact abelian group and $A$ be a commutative semisimple Banach algebra having RNP. Then $M(G,A)$ is a measure algebra only if $A$ is a measure algebra.

PROOF. The theorem follows from Theorem 3.4 and the fact that the algebras $M(G) \Theta \alpha A$ and $M(G,A)$ are isometrically isomorphic under the hypothesis of our theorem.
REFERENCES


16. KAKUTANI, S. Concrete representations of abstract (L)-spaces and the mean ergodic theorem, Annals of Math. (2) 42 (1941), 523-537.
