ON THE ASYMPTOTIC BIEBERBACH CONJECTURE

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ABSTRACT. The set $S$ consists of complex functions $f$, univalent in the open unit disk, with $f(0) = f'(0) - 1 = 0$. We use the asymptotic behavior of the positive semidefinite FitzGerald matrix to show that there is an absolute constant $N_0$ such that, for any $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ with $|a_3| \leq 2.58$, we have $|a_n| < n$ for all $n > N_0$.

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1. INTRODUCTION.

Let $S$ denote the class of all normalized univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disc $D$. The Bieberbach conjecture states that, for functions in $S$, one has $|a_n| \leq n$ for all $n \in \mathbb{N}$. It is known to be true for $n \leq 6$. The best known estimate for all coefficients is $|a_n| \leq (1.066)n$ (Horowitz [1]).

On the other hand, Hayman's Regularity Theorem (Hayman [2]) states that $\lim_{n \to \infty} \frac{|a_n|}{n} \leq 1$ for each $f \in S$, and that equality holds only for the Koebe function $K(z) = \frac{z}{(1-\eta z)^2}$, $|\eta| = 1$, for which $|a_n| = n$. This implies that $|a_n| \leq n$ for $n \geq n_0(f)$.

Hayman [3] also proved that $\frac{A_n}{n}$ tends to a limit, where $A_n$ is the maximum of $|a_n|$ for all $f \in S$. It is still an open question as to whether this limit is equal to 1. The asymptotic Bieberbach conjecture asserts that $\lim_{n \to \infty} \frac{A_n}{n} = 1$, where $A_n = \max_{f \in S} |a_n|$.
Ehrig [4] has proved via the FitzGerald Inequality [5] that if \( f \in S \) and
\(|a_3| \leq C < 2.43\), then \(|a_n| < n\) for all \( n \geq N_0 \), where \( N_0 \) depends only on \( C \) and not
(as in Hayman's Regularity Theorem) on \( f \). This result is a proof of the Asymptotic
Bieberbach Conjecture for a subclass of \( S \).

In this paper, we apply the Asymptotic FitzGerald Inequalities to get, by ele-
mentary means, an improvement of Ehrig's result (Theorem 1) and the result in [6],
(see Remark 2).

2. PRELIMINARY RESULTS.

THEOREM A. (FitzGerald Inequality, [5]). Let
\[ f(z) = z + \sum_{k=2}^{\infty} a_k(f) z^k \]
be in \( S \) and define
\[ q_{mn}(f) = q_{nm}(f) = \left( \sum_{j=1}^{n+m-1} \beta_j(m,n) b_j^2(f) \right) - b_m^2(f) b_n^2(f) \]
where \( b_j(f) = |a_j(f)| \); \( \beta_j(m,n) = \beta_j(n,m) \), \( j \in \mathbb{N} \), and for \( m < n \):
\[ \beta_j(m,n) = \begin{cases} \frac{m-j-n}{|j-n|} & \text{for } |j-n| < m \\ 0 & \text{if otherwise.} \end{cases} \]

Then the FitzGerald matrix
\[ Q(f) = (q_{mn}(f))_{m,n \in \mathbb{N}} \]
is positive semi-definite.

THEOREM B. (Asymptotic FitzGerald Inequalities [7]). Let \( \{f_n\}, n \in \mathbb{N} \), be a
sequence of functions in \( S \), such that
a) \( f_n \) converges locally uniformly to \( f \in S \)
b) \( \lim \inf b_n(f_n)/n \leq \beta \leq \lim \sup b_n(f_n)/n \)
c) \( \alpha(f) = \lim b_n(f)/n \)
d) \( d = \lim \alpha(f_n) \).

Then \( A = Q(j_1, j_2, \ldots, j_{r-1}, \alpha(f), \ldots, \alpha(j), \beta, d, \ldots, d)(f) \), defined below, is a
positive semi-definite matrix.

Denote by \( E_{mn} \) the \( m \times n \) matrix whose elements are all equal to one. Moreover,
let $H_{mn}(f)$ be the $m \times n$ matrix defined by its elements $h_{st}(f) = j_{st}^2 = b_{st}^2(f)$. We use the notation

$$Q(j_1, \ldots , j_{r-1})(f) = (q_{j_s j_t}(f))_{1 \leq s, t \leq p}.$$

where

$$m_{st}(x) = \begin{cases} 
7x^2/6 - x^4 & \text{for } s = t \\
x^2(1 - x^2) & \text{for } s \neq t 
\end{cases}$$

and $\delta = \lim_{n \to \infty} \sup \delta_n$ where $\delta_n = \sup_k b_{kn}(f_k)/n$. Then matrix $A$ has the following form:

$$A = \begin{bmatrix} 
Q(j_1, \ldots , j_{r-1})(f), \alpha(f)H_{r-1, q-r}(f), \beta^2H_{r-1, l}(f), d^2H_{r-1, p-q}(f) \\
\alpha(f)H^T_{r-1, q-r}(f), M_{q-r}(\alpha(f)), \beta^2(1-\alpha^2(f))E_{l-1, 1}, d^2(1-\alpha^2(f))E_{l-1, p-q} \\
\beta^2H_{r-1, l}(f), \beta^2(1-\alpha^2(f))E_{l, q-r}, (7\delta^2/6 - \beta^4)E_{1, 1}, d^2(1-\beta^2)E_{l, p-q} \\
d^2H^T_{r-1, p-q}(f), d^2(1-\alpha^2(j))E_{p-q, q-r}, d^2(1-\beta^2)E_{p-q, l}, M_{p-q}(d) 
\end{bmatrix}$$

where $H^T$ is the transposed matrix of $H$.

**THEOREM C.** [6]. Let $f \in S$; if $|a_3| \leq 2.042$, then $|a_n| < n$ for all $n \geq 2$.

3. **MAIN RESULTS.**

For the proof of the Theorem 1, we need the following lemmas:

**LEMMA 1.** Suppose that $n > 1$ and that

$$\alpha_n(H) = \sup_{f \in H} |a_n|,$$

where $H$ is a compact subclass of $S$. Let $f(z) = z + a_2z^2 + \ldots$ be in $H$ with $|a_n| = \alpha_n(H)$. Then $f(z)$ satisfies the differential equation

$$z^2(f'(z))^2 \frac{1}{a_n} \sum_{v=2}^{n} a_n(v) f(z)^{-v-1} = n-1 + \sum_{v=1}^{n-1} (\frac{va_n}{a_n} z^{-n-v} + \frac{va_n}{a_n} z^{n-v}). \quad (3.1)$$

Here, $a_n(v)$ are the coefficients of $f(z)^v$, where

$$f(z)^v = \sum_{n=v}^{\infty} a_n(v) z^n.$$

The proof of this lemma is completely similar to that of Theorem 1 in Schaeffer and Spencer ([8], p. 612).
As an application of the Lemma 1, we have the following:

**Lemma 2.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathbb{H} \) is the extremal function maximizing \( |a_3| \) such that \( a_3 > 0 \), then \( 2a_3 = a_2^2 + 2 \).

The proof of this lemma is completely similar to that in Garabedian and Schiffer ([9], p. 118).

Hayman [2] showed that for each \( f \in S \), the limits

\[
\alpha(f) = \lim_{r \to 1} (1-r)^2 M_\infty(r,f) = \lim_{n \to \infty} \frac{|a_n(f)|}{n}
\]

exist, where \( M_\infty(r,f) \) is the maximum of \( |f(z)| \) on \( |z| = r \). The number \( \alpha(0 \leq \alpha \leq 1) \) is called the Hayman index of \( f \).

**Lemma 3.** [10]. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \), and \( |a_2| \) is given, then

\[
\alpha(f) = \lim_{n \to \infty} \frac{|a_n(f)|}{n} \leq 4b^{-2} \exp\left(2-4b^{-1}\right)
\]

where \( b = 2 - (2 - |a_2|)^{1/2} \), and this inequality is sharp for \( 0 \leq |a_2| \leq 2 \).

**Lemma 4.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) satisfies the conditions of the Lemma 2 with \( |a_3| > 1 \), then

\[
\alpha(f) = \lim_{n \to \infty} \frac{|a_n(f)|}{n} \leq 4C^{-2} \exp\left(2 - 4C^{-1}\right)
\]

where \( C = 2 - \left[ 2 - \sqrt{2(|a_3| - 1)^{1/2}} \right]^{1/2} \).

**Proof.** By Lemma 3, we have

\[
\alpha(f) \leq 4b^{-2} \exp\left(2 - 4b^{-1}\right)
\]

where \( b = 2 - (2 - |a_2|)^{1/2} \). Since we may assume \( a_3 \) real positive (otherwise, we consider \( e^{-i\theta}f(e^{i\theta}z) \in \mathbb{H} \), where \( 0 \leq \theta = -\frac{\text{arg } a_3}{2} \leq 2\pi \)), we obtain that

\[
b = 2 - (2 - |a_2|)^{1/2} = 2 - \left[ 2 - \sqrt{2(|a_3| - 1)^{1/2}} \right]^{1/2}
\]

\[
= 2 - \left[ 2 - \sqrt{2(|a_3| - 1)^{1/2}} \right]^{1/2} = c.
\]

Hence,

\[
\alpha(f) \leq 4c^{-2} \exp\left(2 - 4c^{-1}\right).
\]

**Lemma 5.** [7]. Let \( \{f_n\}, n \in N \), be a sequence of univalent functions in \( S \),
that converges locally uniformly to a function \( f \) in \( S \) and suppose that \( \alpha(f) > 0 \). Then \( 7\delta^2 \alpha^2(f) \geq 6\delta^4 \), where \( \beta \) and \( \delta \) are chosen as in theorem B.

**Proof.** Consider the \((q - r + 1) \times (q - r + 1)\) principal minor

\[
Q(\alpha(f), \ldots, \alpha(f), \beta) = 
\begin{bmatrix}
M_{q-r}(\alpha(f)) & \beta^2(1-\alpha^2(f))E_{q-r,1} \\
\beta^2(1-\alpha^2(f))E_{1,q-r} & (7\delta^2/6-\beta^4)E_{1,1}
\end{bmatrix}
\]

of the matrix \( A \) in theorem B. A well-known result about positive semidefinite quadratic form is that all principal minor determinants of the matrix of the coefficients of the quadratic form are non-negative. Let \( \alpha = \alpha(f) \) and \( n = q-r \). If we use induction, we obtain:

\[
\text{Det} \ Q(\alpha, \ldots, \alpha, \beta) = (\alpha^2/6)^n (1-\alpha^2) \left[ n(7\delta^2-6\delta^4/\alpha^2) \right] + \alpha^2 n^{-n+1} (7\delta^2-6\delta^4/\alpha^2) \geq 0;
\]

hence, for \( 0 < \alpha < 1 \)

\[
(7\delta^2 - 6\delta^4) + 6(1-\alpha^2)\beta^4
\]

\[
\frac{6\alpha^2}{6n(1-\alpha^2)+1} 
\geq 0.
\]

Since \( n \) is arbitrary, the result follows. The case \( \alpha = 1 \) is immediate.

**Theorem 1.** Let \( f(z) = z + \sum_{k=1}^{\infty} a_k z^k \) be in \( S \). If \( 1 \leq |a_3| \leq 2.58 \), then there is an absolute constant \( N_0 \) (independent of \( f \)), such that \( |a_n| < n \) for all \( n > N_0 \).

**Proof.** Suppose the contrary and take a sequence \( \{g_k\} \), \( k \in N \), of univalent functions in \( S \) such that

i) \( \{g_k\}, k \in N, \) converges locally uniformly to a function \( g_0 \in S \),

ii) \( 1 \leq b_3(g_k) = |a_3(g_k)| \leq 2.58, \)

iii) \( 2a_3(g_k) = a_2(g_k)^2 + 2, \)

iv) \( b_n(g_k) \geq n_k \) for sequence \( n_k \) going to infinity.

**Remark.** The functions \( g_k \) are the extremal functions maximizing \( |a_3(g)| \) in the compact subclass \( H_k = \{g \in S; 1 \leq b_3(g) \leq 2.58 \text{ and } b_n(g) \geq n_k\} \) of \( S \). Applying Lemma 2 to the subclass \( H_k \), we obtain condition (iii).

We pick for each \( n_k \) one of the functions of

\[
\{g_j\}, \ j = 0,1,\ldots,
\]

which maximizes \( b_n \) and denote it by \( f_{n_k} \); precisely, let \( \{f_{n_k}\}, k \in N \), be a sequence
of the functions in
\[ \{g_j\}, j = 0, 1, \ldots, \]
such that
\[ \sup_{j} b_{n_k} (g_j) = b_{n_k} (f_{n_k}). \]
We may assume that \( \{f_{n_k}\}, k \in \mathbb{N} \), converges locally uniformly to a function \( f \in S \).
Otherwise, we pick a subsequence of \( \{f_{n_k}\}, k \in \mathbb{N} \). Evidently, \( 1 \leq b_3(f) \leq 2.58 \) and \( 2a_3(f) = a_2(f)^2 + 2 \). For this sequence \( \{f_{n_k}\}, k \in \mathbb{N} \), we have
\[ \delta_{n_k} = \sup_{j} b_{n_k} (f_{n_k})/n_k = b_{n_k} (f_{n_k})/n_k. \]
Thus
\[ \delta = \lim_{k \to \infty} \sup_{n_k} b_{n_k} (f_{n_k})/n_k > 1. \]
We take \( \beta = \delta \) in Theorem B. First we show that \( \alpha(f) > 0 \). In fact, the determinant of the \( 2 \times 2 \) submatrix \( Q(\alpha(f), \delta) \) of \( A = Q(j_1, \ldots, j_{r-1}, \alpha(f), \ldots, \alpha(f), \beta, d, \ldots, d)(f) \) is
\[ (7\alpha^2(f) - 6\delta^4(f)) \left( \frac{7\delta^2 - 6\delta^4}{36} - \delta^4(1 - \alpha^2(f))^2 \right) \geq 0. \]
This excludes \( \alpha(f) = 0 \) because \( \delta \geq 1 \). By Lemma 5, we have
\[ 7\alpha^2(f)\delta^2 - 6\delta^4 \geq 0 \quad \text{or} \quad \alpha^2(f) \geq \frac{6\delta^2}{7} = \frac{6}{7}. \]
This implies, by Lemma 4, that \( b_3(f) > 2.58 \) which contradicts the assumptions.

**COROLLARY.** Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) be in \( S \). If \( |a_3| \leq 2.58 \), then there is an absolute constant \( N_0 \) (independent of \( f \)), such that \( |a_n| / n \) for all \( n > N_0 \).

**PROOF.** The proof of corollary follows immediately from Theorem 1 and Theorem C.

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**REFERENCES**


6. ALVES, MAURISO. Bieberbach’s Conjecture with $|a_3|$ Restricted – Notas e Comunicações de Matemática, Universidade Federal de Pernambuco, Brazil, (1978).


