A SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATION

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ABSTRACT. The representation of the Hardy-Lebesque space by means of the shift operator is used to prove an existence theorem for a singular functional-differential equation which yields, as a corollary, the well known theory of Frobenius for second order differential equations.

KEY WORDS AND PHRASES. Singular functional-differential equation, Hardy-Lebesque space, Shift-operator.

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1. INTRODUCTION.

Consider the singular functional-differential equation

\[ z^2y''(z) + p(z)y'(z) + q(z)y(z) + \sum_{i=1}^{m} a_i(z)y(q^i z) = 0, \quad |q| \leq 1 \quad (1.1) \]

where

\[ p(z) = \sum_{n=0}^{\infty} a_n z^n, \quad q(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad a_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j, \quad i = 1, 2, \ldots, m \]

are analytic functions in some neighborhood of the closed unit disk \( \Delta = \{ z \in \mathbb{C}; |z| \leq 1 \} \).

We consider the problem of finding conditions for Equation (1.1) to have solutions in the space \( H_2(\Delta) \), i.e. the Hilbert space of functions \( f(z) = \sum_{n=1}^{\infty} a(n) z^{n-1} \) which are analytic in the open unit disk \( \delta = \{ z \in \mathbb{C}; |z| < 1 \} \) and satisfy the condition \( \sum_{n=1}^{\infty} |a(n)|^2 < +\infty \). We shall prove the following.
THEOREM. Let

$$k(k - 1) + a_0k + b_0 = 0$$

(1.2)

be the idicial equation of the unperturbed equation (1.1).

(i) If $2k + a_0 - 1 = \delta = k_1 - k_2 \neq n, n = 1, 2, \ldots$, then Equation (1.1) has two linearly independent solutions of the form:

$$y_1(z) = z^{k_1}u(z) \quad \text{and} \quad y_2(z) = z^{k_2}u(z),$$

where $k_1$ and $k_2$ are the roots of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$.

(ii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = 0$, i.e. $k_1 = k_2$, then Equation (1.1) has only one solution of the form:

$$y(z) = z^{k_1}u(z),$$

where $k$ is the double root of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$.

(iii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = n, n = 1, 2, \ldots$, then Equation (1.1) always a solution of the form:

$$y(z) = z^{k_1}u(z),$$

where $k_1$ is the greatest root of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$.

This theorem obviously generalizes the well known Frobenius theory [1] for the Fuchs differential equations:

$$z^2y''(z) + zp(z)y'(z) + q(z)y(z) = 0,$$

which is a particular case of Equation (1.1).

Denote an abstract separable Hilbert space over the complex field by $H$, the Hardy-Lebesque space by $H_2(\Delta)$, an ortho-normal basis in $H$ by $\{e_n\}_{n=1}^{\infty}$, and the unilateral shift operator on $H (V: V e_n = e_{n+1})$ by $V$. We can easily see that the following statements hold:

(i) Every value $z$ in the unit disk $(|z| < 1)$ is an eigenvalue of $V^*(V^*: V^* e_n = e_{n-1}, n \neq 1, V^* e_1 = 0)$, the adjoint of $V$. The eigenelements

$$f_z = \sum_{n=1}^{\infty} z^{n-1}e_n$$

form a complete system in $H$, in the sense that if $f$ is orthogonal to $f_z$, for every $z$: $|z| < 1$ then $f = 0$.

(ii) The mapping $f(z) = (f_z, f), f \in H$ is an isomorphism from $H$ onto $H_2(\Delta)$. 
(iii) The diagonal operator $C_0: C_0 e_n = n e_n$, $n = 1, 2, \ldots$, has a self-adjointed extension in $H$ with a compact inverse $B: B e_n = \frac{1}{n} e_n$, $n = 1, 2, \ldots$. Moreover, if $f(z) = (f, f)$ then

$$z^n f(z) = (f, V^n f) \quad (1.3)$$

$$f^{(n)}(z) = (f, (C_0 V^*)^n f) \quad (1.4)$$

$$zf'(z) = (f, (C_0 - I) f) \quad (1.5)$$

We shall use the proposition 1 of Reference [2].

2. PROOF OF THE THEOREM.

The transformation $y(z) = z^k u(z)$, reduces Equation (1.1) in the following:

$$zu''(z) + (h_0 + h_1 z + h_2 z^2 + \ldots) u'(z) + (a_0 + a_1 z + a_2 z^2 + \ldots) u(z) + \sum_{i=1}^{m} q^i a_i(z) u(q^i z) = 0, \quad (2.1)$$

where $k(k - 1) + ka_0 + b_0 = 0$, $2k + a_1 = h_0$, $a_2 = h_1$, $a_3 = h_2$, \ldots and $ka_1 + b_1 = 0$, $ka_2 + b_2 = a_1$, $ka_3 + b_3 = a_2$, \ldots. Following Reference [2], we define the operators $R_1, R_2, \ldots, R_m$ on $H_2(\Delta)$ as

$$R_1 u(z) = u(qz), \quad |q| \leq 1, \quad R_2 u(z) = u(q^2 z) = R_1 u(z), \quad \ldots, \quad R_m u(z) = u(q_m z) = R_1^m u(z).$$

Thus the operator $R: Ru(z) = \sum_{i=1}^{m} q^i a_i(z) u(q^i z)$, $|q| \leq 1$, on $H_2(\Delta)$ is represented in the space $H$ by the operator

$$\tilde{R}: \tilde{R} u = \sum_{i=1}^{m} q^i a_i(V) u(q^i z)$$

where $\tilde{R}_1$ is defined on $H$ as $\tilde{R}_1 e_n = q^{n-1} e_n$, $n = 1, 2, \ldots$. The equation (2.1) has a solution in $H_2(\Delta)$ if and only if the operator equation

$$[V(C_0 V^*)^2 + \phi_1(V) C_0 V^* + \phi_2(V) + \tilde{R}] u = 0 \quad (2.2)$$

has a solution $u$ in the abstract separable Hilbert space $H$.

Here $u = \sum_{n=1}^{\infty} (u, e_n) e_n$, $\phi_1(V) = (2k + a_0) I + h_1 V + h_2 V^2 + \ldots$, $\phi_2(V) = \rho_0 I + \rho_1 V + \rho_2 V^2 + \ldots$, where the bar denotes complex conjugation.

Taking into account the relations

$$V^2 C_0 V^* = V(C_0 - I) \quad \text{and} \quad VC_0 - C_0 = -V,$$

Equation (2.2) can be written as
\[ [C_0 + (2k + a_0 - 1)I + B\phi(V) - B^2\psi_1'(V)] V^* + B\phi_2(V) + B_2R \] \[ \text{u} = 0, \quad (2.3) \]

where

\[ \phi(V) = h_1V + h_2V^2 + h_3V^3 + \ldots \quad \text{and} \quad \phi_1'(V) = h_1 + 2h_2V + 3h_3V^2 + \ldots. \]

Also, if we put \(2k + a_0 - 1 = \delta\) in Equation (2.3), we have

\[ V^*[I + VK] \text{u} = 0 \quad (2.4) \]

where the operator

\[ K = 5B\psi_1 + B^2\phi(V)C_0V^* + B^2\phi_2(V) + B^2R \]

is compact. Relation (2.4) implies that

\[ (I + VK) \text{u} = c \text{e}_1, \quad c = \text{const.} \quad (2.5) \]

Now it follows that the operator \((I + VK)^{-1}\) exists. In fact,

\[(I + VK)\text{u} = 0 \Rightarrow \text{u} = -VK\text{u} \Rightarrow (u, e_1) = -(K\text{u}, e_1) = 0. \]

Also,

\[ (u, e_2) = -(u, Ke_1) \Rightarrow (u, e_2)(1 + \delta) = 0 \quad (2.6) \]

Relation (2.6) if \(\delta \neq -1 \Rightarrow (u, e_2) = 0. \) Similarly,

\[ (u, e_3) = -(u, Ke_2) \Rightarrow (u, e_3)(1 + 2\delta) = 0. \quad (2.7) \]

Relation (2.7) if \(\delta \neq -2 \Rightarrow (u, e_3) = 0. \) By the same way and if \(\delta \neq -n, n = 1, 2, \ldots, \)

we find

\[ \text{u} = \sum_{n=1}^{\infty} (u, e_n)e_n = 0. \]

Since also the operator \(VK\) is compact Fredholm alternative implies that the operator \((I + VK)^{-1}\) is defined every where. Thus from Equation (2.5), we have

\[ \text{u} = c \cdot (I + VK)^{-1} e_1. \]

This means that

(i) If \(2k + a_0 - 1 = \delta = k_1 - k_2 \neq \pm n \) with \(n = 1, 2, \ldots, \) then the operator \((I + VK)^{-1}\) always exists. Therefore, Equation (1.1) has two linearly independent solutions of the form

\[ y_1(z) = z^{k_1}u(z) \quad \text{and} \quad y_2(z) = z^{k_2}u(z), \]

where \(k_1\) and \(k_2\) are the roots of Equation (1.2) and \(u(z)\) belongs to \(H_2(\Delta)\) and is given by the relation

\[ u(z) = (u_z, u), \quad u_n = \sum_{n=1}^{\infty} z^{n-1}e_n, \quad \text{u} = c \cdot (I + VK)^{-1} e_1. \]
(ii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = 0$, i.e. $k_1 = k_2$, then the operator $(I + VK)^{-1}$ always exists. Therefore, Equation (1.1) has only one solution of the form

$$y(z) = z^k u(z),$$

where $k$ is the double root of Equation (2.1) and $u(z)$ as in (i).

(iii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = n$, $n = 1, 2, \ldots$, then

$$2k_1 + a_0 - 1 = n, \quad n = 1, 2, \ldots,$$

$$2k_2 + a_0 - 1 = -n, \quad n = 1, 2, \ldots,$$

From the above and the Relations (2.6) and (2.7), we see that Equation (1.1) has always a solution of the form

$$y(z) = z^{k_1} u(z),$$

where $k_1$ is the greatest root of Equation (1.2) and $u(z)$ as in (i). All the above complete the proof of the theorem.

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REFERENCES
