BAZILEVIC FUNCTIONS OF TYPE $\beta$

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ABSTRACT. In this paper, a new coefficient result for the Bazilevič functions of type $\beta$ is obtained.

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1. INTRODUCTION.

Let $S$ denote the class of functions $f$ which are analytic and univalent in $E = \{z: |z| < 1\}$ and which satisfy $f(0) = 0$ and $f'(0) = 1$. Let $S^*$ be the class consisting of starlike functions. Bazilevič [1] introduced a class of analytic functions $f$ defined by the following relation. For $z \in E$, let

$$f(z) = \left\{ \begin{array}{l}
\frac{\beta}{1 + a^2} \left[ \int_0^z \frac{-\beta a_1}{1 + a^2} - 1 - 1 \beta/1 + a^2 g \left( \zeta \right) d\zeta \right]
\end{array} \right\}, \quad (1.1)$$

where $a$ is real, $\beta > 0$, Re$H(z) > 0$ and $g \in S^*$. Such functions, he showed, are univalent [1]. With $a = 0$ in (1.1), we have [2] for $z \in E$,

$$\text{Re} \left\{ \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} \right\} > 0. \quad (1.2)$$

We shall denote this class of functions by $B(\beta)$. We notice that, if $\beta = 1$ in (1.2), we have the class of close-to-convex functions first introduced by Kaplan [3].
2. MAIN RESULTS.

Denote \( M(r,f) = \max_{|z|=r} |f(z)|, \) \( 0 \leq r < 1 \) and \( M(r,f) \leq (1 - r)^{-\alpha}, 0 \leq \alpha \leq 2. \)

Thomas [2] has proved that \( n|a_n| \leq K(\alpha, \beta)^n. \) We improve his result as follows:

**THEOREM 1.** Let \( f \in B(\beta), \) for \( 0 < \beta \leq 1, \) and be given by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \)

Then, for \( n \geq 2, \)

\[
|a_n| \leq A(\beta) \frac{(2n-1)}{2n} \]

where \( A(\beta) \) is a constant depending only upon \( \beta. \)

**PROOF.** From (1.2), we can write

\[
zf'(z) = f^{1-\beta}(z) g^\beta(z) h(z), \quad \text{Re } h(z) > 0; \quad g \in S^* .
\]

Thus,

\[
(zf'(z))' = (1 - \beta) f^{1-\beta}(z) f'(z) g^\beta(z) h(z) + \beta f^{1-\beta}(z) g^{\beta-1}(z) g'(z) h(z) + f^{1-\beta}(z) g^\beta(z) h'(z). \quad (2.4)
\]

Since, with \( z = re^{i\theta}, \) Cauchy's theorem gives

\[
|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| e^{-i\theta} d\theta ,
\]

we have from (2.4),

\[
|a_n| \leq \frac{1}{2\pi r^n} \left\{ (1 - \beta) \int_0^{2\pi} |zf'(z)| f^{-\beta}(z) g^\beta(z) h(z) |d\theta \right. \\
+ \beta \left. \int_0^{2\pi} |zf'(z)| f^{1-\beta}(z) g^{\beta-1}(z) h(z) |d\theta \right. \\
+ \left. \int_0^{2\pi} |f^{1-\beta}(z) g^\beta(z) h'(z) |d\theta \right\} \\
= \frac{1}{r^n} |I_1 + I_2 + I_3|, \text{ say.}
\]

Now,

\[
I_1 = \frac{(1 - \beta)}{2\pi} \int_0^{2\pi} |zf'(z)| f^{-\beta}(z) g^\beta(z) h(z) |d\theta \\
= \frac{(1 - \beta)}{2\pi} \int_0^{2\pi} |f'(z)|^2 |f(z)|^{-\beta} |d\theta, \text{ using (2.3).}
\]

\[
I_2 = \frac{(1 - \beta)}{2\pi} \int_0^{2\pi} |zf'(z)| f^{1-\beta}(z) g^{\beta-1}(z) h(z) |d\theta \\
= \frac{(1 - \beta)}{2\pi} \int_0^{2\pi} |f'(z)|^2 |f(z)|^{\beta-1} |d\theta, \text{ using (2.3).}
\]

\[
I_3 = \frac{1}{2\pi} \int_0^{2\pi} |f^{1-\beta}(z) g^\beta(z) h'(z) |d\theta
\]
In order to estimate this integral, we use the following well-known result [4, p. 46].

Suppose that $f$ is a really mean $p$-valent in $E$ and that $\frac{1}{2} \leq r < 1$, $0 < \lambda < 2$. Then there exists $\rho$ such that $2r - 1 \leq \rho < r$ and

$$
\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \frac{1}{|f(re^{i\theta})|^2} \lambda^{-1} d\theta \leq \frac{4PM(r,f)^\lambda}{\lambda(1-r)}.
$$

(2.5)

With $\rho = 1$ and $\lambda = 1$ in (2.5), we have

$$
\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \frac{1}{|f(re^{i\theta})|^2} \lambda^{-1} d\theta \leq \frac{4M(r,f)}{1-r}.
$$

Since $r < \frac{1+p}{2}$ and $M(r,f)$ is an increasing function, $M(r,f) \leq M\left(\frac{1+p}{2}, f\right)$. Also,

$$
\frac{1}{1-r} \leq \frac{2}{1-\rho} \quad \text{since } 2r - 1 \leq \rho \leq r.
$$

Thus

$$
I_1 \leq 8(1-\beta) \frac{M\left(\frac{1+p}{2}, f\right)}{(1-\rho)}.
$$

Choosing $\rho = 1 - \frac{1}{n}$, we obtain for $n \geq 2$, see [5, p. 238, 240],

$$
I_1 \leq 8(1-\beta) M\left(\frac{2n-1}{2n}, f\right), \quad n.
$$

For $z = re^{i\theta}$,

$$
I_2 = \frac{\beta r}{2\pi} \int_0^{2\pi} |zg'(z)f^{1-\beta}(z)g^{\beta-1}(z)b(z)| d\theta
$$

$$
= \frac{\beta r}{2\pi} \int_0^{2\pi} |f'(z)\varphi(z)| d\theta, \quad \text{where } \varphi(z)g(z) = zg'(z); \quad \text{Re}\varphi(z) > 0.
$$

Applying the Schwarz inequality, we have

$$
I_2 \leq \frac{\beta r}{2\pi} \left( \int_0^{2\pi} |f'(z)|^2 d\theta \right)^{1/2} \left( \int_0^{2\pi} |\varphi(z)|^2 d\theta \right)^{1/2}.
$$

Now

$$
\frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 d\theta = \sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2} \leq \sum_{n=1}^\infty n |a_n|^2 r^n, \quad \text{max } n r^{n-2}.
$$

Since the function $\log (nr^n)$ has a maximum at a point $n_0 = \frac{1}{\log \frac{1}{r}}$, we have

$$
\log nr^n \leq \log n_0 r^{n_0} = \log \frac{1}{e \log \frac{1}{r}}.
$$
\[ i.e., n r n - 2 \leq - \frac{1}{e r^2 \log \frac{1}{r}} \leq \frac{1}{e r^2 (1 - r)} \quad . \]  

(2.6)

Also, it is well-known [6, p. 42] that \( M(r, f) \leq \frac{4}{r} M(r^2, f) \),
and that the area principal for univalent functions gives
\[ A(r, f) \leq \pi M^2(r, f), \text{ (see [7, p. 215]).} \]

(2.7)

(2.8)

Using (2.6), (2.7) and (2.8), we obtain
\[ \frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 d\theta \leq \frac{16}{e} \frac{M(r, f)^2}{r^3 (1 - r)} \quad . \]

(2.9)

Since \( \text{Re } \phi(z) > 0 \),
\[ \phi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(t) = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ with } |c_n| \leq 2 ; \quad \text{ see [4].} \]

Thus
\[ \frac{1}{2\pi} \int_0^{2\pi} |\phi(z)|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq 1 + 4 \sum_{n=1}^{\infty} r^{2n} = 1 + \frac{3r^2}{1 - r^2} \quad . \]

(2.10)

From (2.9) and (2.10), we have
\[ I_2 \leq \frac{4\beta}{\sqrt{e} r} \frac{M(r, f)}{(1 - r)} \left[ 1 + \frac{3r^2}{1 + r} \right] \leq \frac{4\sqrt{2} \beta}{\sqrt{e} r} \frac{M(r, f)}{(1 - r)} \quad . \]

Since \( r \leq \frac{1 + r}{2} \) and \( M(r, f) \) is an increasing function, we have for \( r = \frac{1 - \frac{1}{n}}{n} \) and \( n \geq 2 \)
\[ I_2 \leq \frac{4\sqrt{2} \beta}{\sqrt{e} r} M\left( \frac{2n - 1}{2n} , f \right) . \quad n \]
\[ \leq \frac{8\beta}{\sqrt{e} r} M\left( \frac{2n - 1}{2n} , f \right) . \quad n \]

Finally, since \( 0 < \beta \leq 1, z = re^{i\theta} \) and
\[ I_3 = \frac{1}{2\pi} \int_0^{2\pi} |f^{1-\beta}(z)g^{\beta}(z) z h'(z)| d\theta \]
\[ \leq M^{1-\beta}(r, f) \cdot \frac{1}{2\pi} \int_0^{2\pi} |g^{\beta}(z) z h'(z)| d\theta \]
\[ \leq M^{1-\beta}(r, f) \cdot \frac{2r}{1 - r^2} \frac{1}{2\pi} \int_0^{2\pi} |g^{\beta}(z)| \text{Re} h(z) d\theta . \]

Since it is known [8] that \( |z h'(z)| \leq \frac{2r \text{Re} h(z)}{1 - r^2} \), then by (3)
Integrating by parts and using the fact that \( g \) is starlike, we obtain

\[
I_3 \leq M^{1-\beta}(r,f) \cdot \frac{2r}{1 - r^2} \left[ \int_0^{2\pi} \beta e^{1-\beta} f(z) f'(z) e^{-i\arg g(z)} d\theta \right]
\]

Thus, for \( n \geq 2 \)

\[
n \left| a_n \right| \leq e \left\{ 8(1 - \beta) + \frac{8\beta}{\lambda e} + \beta \right\} M\left( \frac{2n - 1}{2n} , f \right), \quad (\text{see } [4, \text{p. 45}]).
\]

This completes the proof.

REFERENCES


