POSITIVE SOLUTIONS OF THE DIOPHANTINE EQUATION

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ABSTRACT. Integral solutions of \( x^3 + \lambda y + 1 - xyz = 0 \) are observed for all integral \( \lambda \). For \( \lambda = 2 \) the 13 solutions of the equation in positive integers are determined. Solutions of the equation in positive integers were previously determined for the case \( \lambda = 1 \).

KEY WORDS AND PHRASES. Diophantine equation, cubic, positive solution.

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1. INTRODUCTION.

The Diophantine equation

\[ x^3 + \lambda y + 1 - xyz = 0 \]  \hspace{1cm} (1)

is always satisfied by the positive triple \((2\lambda + 1, 2, 2\lambda^2 + 2\lambda + 1)\). For \( \lambda = 1 \), S. P. Mohanty [1] has given all 9 positive solutions of this equation and in a sequel [2] has given all integral solutions of this equation. In this paper we determine all of the 13 positive solutions of

\[ x^3 + 2y + 1 - xyz = 0. \]  \hspace{1cm} (2)

Equation (2) has an infinite number of integral solutions. For example, \((-1,0,z), (-1,y,-2)\) are solutions of (2). In general \((-1, 0, z)\) and \((-1, y, -\lambda)\) satisfy (1).

THEOREM. There are only a finite number of solutions of (2) in positive integers.

PROOF. As in [1] we write the given equation (2) as an equivalent system. If \((x,y,z)\) satisfies (2), then \(x|2y + 1\) and \(y|x^3 + 1\). Conversely, if \(x, y\) are positive integers for which \(x|2y + 1\) and \(y|x^3 + 1\), then \(xy|x^3 + 2y + 1\) hence for some positive \(z\)
one has $x^3 + 2y + 1 - xyz = 0$.

Hereafter, we focus attention on the system $x|2y + 1$, $y|x^3 + 1$. If $(x, y)$ are positive integers for which these statements prevail, then there are positive integers $r, s$ for which

$$
rx = 2y + 1 \quad (3)
$$

$$
sy = x^3 + 1 \quad (4)
$$

Eliminating $y$ from (3), (4), one has

$$
s(rx - 1) = 2x^3 + 2 \quad (5)
$$

which may be written as

$$
x(sr - 2x^2) = s + 2 \quad (6)
$$

Let $n = sr - 2x^2$, a positive integer, to secure $xn = s + 2$ from (6).

Then

$$
2x^2 = sr - n = (xn - 2) - n = rnx - (2r + n). \quad (7)
$$

The extremes of this equation imply $2x < rn$ from which we gain the existence of a positive integer for which

$$
rn = 2x + k \quad (8)
$$

Combining (7), (8) we have

$$
xk = 2r + n \quad (9)
$$

and finally, that

$$(n - 2)(r - 1) + (x - 1)(k - 2) = 4. \quad (10)$$

If we write

$$
A = (n - 2)(r - 1)
$$

$$
B = (x - 1)(k - 2)
$$

then (10) becomes $A + B = 4$. We continue the proof by considering the cases $A < 0, B < 0, A = 0, B = 0,$ and then the case where $A, B$ are both positive.

Case $A < 0$. For this case, $n = 1$ and $B > 0$ (in particular, $k > 2$).

From (10),

$$
x = 1 + \frac{r + 3}{k - 2} \quad (11)
$$

From (8), with $n = 1$,

$$
x = 1 + \frac{2x + k + 3}{k - 2}$$
and hence
\[ x = \frac{2k + 1}{k - 4} = 2 + \frac{9}{k - 4} \]
Thus, \( k - 4 \mid 9 \) and \( k = 5, 7, 13, 3, 1, -5 \). For \( k = -5 \), \( y \) is negative; for \( k = 1, 3 \), \( x \) is negative. Given \( k \), \( x = \frac{2k + 1}{k - 4} \), \( r = 2x + k \) and \( y = \frac{rx - 1}{2} \).
Starting this sequence with \( k = 5, 7, 13 \) one secures \( (x, y) = (5, 42), (11, 148), (3, 28) \), respectively.

Case B < 0. This case implies \( k = 1 \) and \( A > 0 \) (in particular, \( n > 2 \)). For \( k = 1 \), (8), (9) becomes \( rn = 2x + 1 \) and \( x = 2r + n \). If we eliminate \( n \) from these equations, we secure
\[ (r - 2) x = 2r^2 + 1 \]
(12)
The case \( r = 1 \) is included below (Case A = 0). \( r = 2 \) implies \( x = 4 + n \) by (9). Since \( 2y = rx - 1 = 2n + 7 \), \( y \) is not an integer and so no solution results from \( r = 2 \). We now consider \( r > 2 \) and write
\[ x = \frac{2r^2 + 1}{r - 2} = 2r + 4 + \frac{9}{r - 2} \]
from which we infer that \( r = 3, 5, 11, 1, -1, -7 \). For the last three values, \( x < 0 \). For \( r = 3, 5, 11 \) we calculate \( x = \frac{(2r^2 + 1)}{(r - 2)} \), \( y = \frac{(rx - 1)}{2} \) to secure, respectively, the pairs \( (x, y) = (19, 28), (17, 42), (27, 148) \).

Case A = 0. In this case, \( B = 4 \). Since \( B = 4 \), \( (x, k) = (2, 6), (3, 4), (5, 3) \). Since \( A = 0 \), either \( r = 1 \) or \( n = 2 \). If \( r = 1 \) we recall that \( 2y = x - 1 \) (from (3)) hence \( (x, y) = (3, 1), (5, 2) \) result as solutions (\( x = 2 \) does not give an integral \( y \)). If \( n = 2 \) we compute \( r \) from \( 2r = 2x + k \) (equation (8)) and then compute \( y \) from \( 2y = rx - 1 \) to secure one usable \( r(=5) \) from which the solution \( (x, y) = (3, 7) \) results.

Case B = 0. This case is similar to \( A = 0 \) and gives three pairs \( (x, y) = (5, 7), (1, 2), (1, 1) \).

Case A > 0 and B > 0. This gives three subcases. (a) \( A = 1, B = 3 \);
(b) \( A = B = 2 \); (c) \( A = 3, B = 1 \). Clearly, these cases yield a finite number of solutions since, in particular, \( x \) and \( r \) are bounded and, because of (3), \( y \) may be determined from them.
For (a) we have \((n - 2)(x - 1) = 1, (x - 1)(k - 2) = 3\). Thus, \(n = 3\) and \(r = 2\).

None of the possible pairs \((x, k) = (2, 5), (4, 3)\) gives an integral \(y\).

For (b), \((n - 2)(r - 1) = (x - 1)(k - 2) = 2\). Thus \((n, r) = (4, 2), (3, 3)\) and \((x, k) = (2, 4), (3, 3)\). The pair \(r = 3, x = 3\) yields the only solution, \((x, y) = (3, 4)\).

Similarly, for (c) one secures no solution.

This concludes the proof of the theorem.

We conclude by giving the complete set of positive triples \((x, y, z)\) for which (2) is satisfied: \((1, 1, 4), (1, 2, 3), (3, 1, 10), (3, 4, 3), (3, 7, 3), (3, 28, 1), (5, 2, 13), (5, 7, 4), (5, 42, 1), (11, 148, 1), (17, 42, 7), (19, 28, 13), (27, 148, 5)\).

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REFERENCES


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