A FIXED POINT THEOREM FOR CONTRACTION MAPPINGS

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ABSTRACT. Let S be a closed subset of a Banach space E and f: S → E be a strict contraction mapping. Suppose there exists a mapping h: S → (0,1] such that 
\((1 - h(x))x + h(x)f(x) \in S\) for each \(x \in S\). Then for any \(x_0 \in S\), the sequence \(\{x_n\}\) in S defined by 
\(x_{n+1} = (1 - h(x_n))x_n + h(x_n)f(x_n), n \geq 0,\) converges to a \(u \in S\). Further, if \(\sum h(x_n) = \infty\), then \(f(u) = u\).

KEY WORDS AND PHRASES. Contraction mapping

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1. INTRODUCTION.

In a recent paper [1], Ishikawa proved the following result.

THEOREM. Let S be a closed subset of a Banach space E and let f be a nonexpanive mapping from S into a compact subset of E. Suppose there exists a real sequence \(\{h_n\}, 0 < h_n < b < 1\) and an \(x_0 \in S\) such that \(x_{n+1} = (1 - h_n) x_n + h_n f x_n \in S\) for each \(n > 0\). If \(\sum h_n = \infty\), then the sequence \(\{x_n\}\) converges to a fixed point of f.

In this note, we investigate the above result when f therein is a contraction mapping (for some \(\alpha, 0 < \alpha < 1, \| f x - f y \| \leq \alpha \| x - y \|,\) for all \(x, y \in S\) but does not necessarily have a precompact range. We show that if \(0 < h_n < 1\), then the sequence \(\{x_n\}\) above converges to a \(u \in S\) and if \(\sum h_n = \infty\) then \(f u = u\). The proof is much less computational in this case.

2. MAIN RESULT.

Throughout, let E denote a Banach space. The main result is
THEOREM 1. Let $S$ be a closed subset of $E$ and $f: S \to E$ be a contraction mapping satisfying the condition: there exists a mapping $h: S \to (0,1]$ such that for each $x \in S$,

$$(1 - h(x))x + h(x)f(x) \in S. \quad (1.1)$$

If $x_0 \in S$ and the sequence $\{x_n\}$ in $S$ is defined by

$$x_{n+1} = (1 - h(x_n))x_n + h(x_n)f(x_n), \quad n \geq 0, \quad (1.2)$$

then (a) the sequence $\{x_n\}$ converges to a $u \in S$ and (b) if $\sum h(x_n) = \infty$, then $u$ in (a) is the unique fixed point of $f$.

The following result (see Knopp [2], Theorem 4, p. 220) is used in the proof of Theorem 1.

PROPOSITION 1. Let $\{a_n\}$ be a sequence of reals with $0 < a_n < 1$. Then the sequence $\{\sum_{i=1}^{n} (1 - a_i)\} \to b > 0$ iff $\sum a_n = \infty$.

Proof of Theorem 1. Let $h_n = h(x_n)$. It follows by (2) that

$$x_{n+1} - x_n = h_n(fx_n - x_n), \quad (1.3)$$

and $fx_n - x_{n+1} = (1 - h_n)(fx_n - x_n). \quad (1.4)$

Thus, for each positive integer $n$,

$$|| f_{x_n} - x_n || \leq || f_{x_n} - f_{x_{n-1}} || + || f_{x_{n-1}} - x_n || \leq a || x_n - x_{n-1} || + (1 - h_{n-1}) || f_{x_{n-1}} - x_{n-1} ||. \quad (1.5)$$

Therefore, it follows by (1.3) that

$$|| f_{x_n} - x_n || \leq \frac{1}{a} || f_{x_0} - x_0 || \quad \text{and} \quad \sum_{i=0}^{n-1} (1 - (1 - a)h_i) || f_{x_0} - x_0 || \leq \sum_{i=0}^{n} || f_{x_i} - x_i || \leq \infty. \quad (1.5)$$

Set $u_i = (1 - a)h_i$. Since $0 < u_i < 1$, $\{\sum_{i=0}^{n} (1 - u_i)\}$ is a decreasing sequence of positive reals and hence there is a $b > 0$ such that $\sum_{i=0}^{n} (1 - u_i) = b$. We consider two cases (i) $b > 0$ and (ii) $b = 0$. If $b > 0$, then by Proposition 1,

$$\sum (1 - a)h_i < \infty \quad \text{and hence} \quad \sum h_i < \infty. \quad \text{Consequently, by (1.3) and (1.5),} \quad \sum || x_{n+1} - x_n || \leq || f_{x_0} - x_0 || \sum h_n < \infty.$$
This implies that the sequence \( \{x_n\} \) is a Cauchy sequence in \( S \) and hence there is an \( u \in S \) such that \( \{x_n\} \to u \). Thus (a) holds in this case. If \( b = 0 \) then it follows by (1.5) that
\[
|| x_n - f x_n || \to 0. \tag{1.6}
\]
Since for any \( m \geq n \),
\[
|| x_m - x_n || \leq || x_m - f x_m || + || f x_m - f x_n || + || f x_n - x_n ||
\]
\[
\leq a || x_m - x_n || + 2 || x_n - f x_n ||,
\]
it follows that \( || x_m - x_n || \leq 2(1 - a)^{-1} || x_n - f x_n || \to 0 \) as \( n \to \infty \). Thus \( \{x_n\} \) is a Cauchy sequence and hence converges to \( u \in S \). Furthermore, it follows by (1.6) that \( u = f u \). This establishes (a). Now, if \( h(x) = -1 \) then \( \sum h(x_n) = \infty \) and hence by Proposition 1, \( b = \lim_{i=0}^m (1 - u_i) = 0 \). Consequently, by case (ii) the sequence \( \{x_n\} \to u \) and \( f u = u \). The uniqueness is obvious for such mappings.

For \( x, y \in E \), let \( [x,y] = \{ z \in E : z = (1 - h)x + hy, 0 < h < 1 \} \). Let \( (x,y) = [x,y] \setminus \{x,y\} \). As an application of Theorem 1, we have

COROLLARY 1. Let \( S \) be a closed subset of \( E \) and \( f : S \to E \) be a contraction mapping. If for each \( x \in S \), there exists a \( y \in [x,fx] \cap S \) such that \( fy \in S \), then \( f \) has a fixed point.

PROOF. Define \( h : S \to (0,1) \) as follows. If \( fx \in S \), let \( h(x) = 1 \) and if \( fx \notin S \), then choose a \( y \in [x,fx] \cap S \) with \( fy \in S \) (such a \( y \) exists by hypothesis). Clearly, \( y \neq x \) and \( y = (1 - h)x + h f x \) for some \( h \) with \( 0 < h < 1 \). Let \( h(x) = h \) in this case. Thus (1.1) holds. Note that if \( f(x) \notin S \) then \( h(y) = 1 \). Now, for any \( x_0 \in S \) and the sequence \( \{x_n\} \) defined by (1.2) that is,
\[
x_{n+1} = (1 - h(x_n))x_n + h(x_n)f(x_n), \quad \text{either } h(x_n) = 1 \text{ or } h(x_{n+1}) = 1 \text{ according as } f x_n \in S \text{ or } f x_n \notin S.
\]
In either case \( \sum h(x_n) = \infty \). Thus by Theorem 1, \( f \) has a fixed point.

It is known (see [3]) that if \( S \) is a closed subset of \( E \) and \( x, y \in E \) such that \( x \) is an interior point of \( S \) and \( y \notin S \), then there \( z \in (x,y) \cap \partial S \). As a consequence of this result and Corollary 1, we have

COROLLARY 2. Let \( S \) be a closed subset of \( E \) and \( f : S \to E \) be a contraction mapping. If \( f(\partial S) \subseteq S \) then \( f \) has a fixed point.

PROOF. If for \( x \in S \), \( f x \in S \), then \( y = x \) satisfies the condition in Corollary
1 and if $fx \notin S$ then by hypothesis $x \notin SS$. Consequently, there is a $y \in (x, fx) \cap SS$ with $fy \in S$. Thus by Corollary 1, $f$ has a fixed point.

We now give two examples. Example 1 shows that Corollary 2 is indeed a special case of Theorem 1. In Example 2, we show that if $\sum h(x_n) < \infty$ in Theorem 1, then the sequence $\{x_n\}$ may not converge to a fixed point.

**EXAMPLE 1.** Let $S = \{0, 2^{-n}: n \geq 0\}$. Define a mapping $f: S \to \mathbb{R}$ (reals) by

$$f(2^{-n}) = 3 \cdot 2^{-(n+3)}, \quad n \geq 0,$$

$$f(0) = 0.$$

It is clear that any $x, y \in S$, $\|fx - fy\| \leq (3/8)\|x - y\|$. Let $h: S \to (0, 1]$ be defined by $h(0) = 1$ and $h(x) = (4/5)$ for $x \neq 0$. It is easy to verify that for $x = 2^{-n}$, $(1 - h(x))x + h(x)f(x) = 2^{-(n+1)}$, while for $x = 0$, it is clearly 0. Thus (1.1) holds. Further, if $x_0 = 1$, then by (1.2), $x_n = 2^{-n}$ and since $\sum h(x_n) = \infty$, Theorem 1 implies the existence of a $u \in S$ with $fu = u$ (which is 0 in this case).

Note that $f(S)$ is not a subset of $S$.

**EXAMPLE 2.** Let $\{a_n\}$ be a sequence of reals defined by $a_1 = 1$ and

$$a_n = \frac{1}{2} \left(1 - 2^{-i}\right) \text{ for } n \geq 2.$$

Since $\sum 2^{-i} < \infty$, it follows by Proposition 1 that $\{a_n\} + b > 0$. Let

$$S = [0, b] \cup \{a_n: n \geq 1\}.$$

Let $fx = 2^{-1}x$ for each $x \in S$. Define $h: S \to (0, 1]$ by

$$h(x) = 1 \text{ if } x \in [0, b]$$

$$= 2^{-n}, \quad \text{if } x = a_n, \quad n \geq 1.$$

Then for any $n \geq 1$, $a_{n+1} = (1 - h(a_n))a_n + h(a_n)f(a_n)$. Since $f[0, b] \subseteq [0, b]$, it follows that $f$ satisfies (1.1). Also, if $x_0 = 1$, and the sequence $\{x_n\}$ is as constructed in (1.2), then $x_n = a_n$ and $\{x_n\} \to b$ but $f(b) \neq b$. Note that $\sum h(x_n) = \sum (x_n^{-1}) < \infty$ in this case.

**REFERENCES**

