STABILITY IMPLICATIONS ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SYSTEMS

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ABSTRACT. In this paper we generalize Bownds' Theorems (1) to the systems \( \frac{dY(t)}{dt} = A(t)Y(t) \) and \( \frac{dX(t)}{dt} = A(t)X(t) + F(t,X(t)) \). Moreover, we also show that there always exists a solution \( X(t) \) of \( \frac{dX}{dt} = A(t)X + B(t) \) for which \( \lim_{t \to \infty} \sup \|X(t)\| > 0 \) if there exists a solution \( Y(t) \) for which \( \lim_{t \to \infty} \sup \|Y(t)\| > 0 \).

KEY WORDS AND PHRASES. stable, norm, linear systems, null solution, Schauder-Tychonoff Theorem, uniformly converges, equicontinuous.

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1. INTRODUCTION.

In this paper we shall study the stability behavior of the following systems

\[ \frac{dY(t)}{dt} = A(t)Y(t), \quad 0 \leq t < \infty \] (1.1)

and

\[ \frac{dX(t)}{dt} = A(t)X(t) + F(t,X(t)), \quad 0 \leq t < \infty \] (1.2)

where \( A(t) \) is a continuous matrix on \( \mathbb{R}^n \) for all \( 0 \leq t < \infty \), \( F(t,X(t)) \) is a real valued continuous \( n \)-vector defined on \( [0,\infty) \times \mathbb{R}^n \) and \( X(t) \) and \( Y(t) \) are \( n \)-vectors.

Consider special equations of (1.1) and (1.2)

\[ y'' + a(t)y = 0, \quad 0 \leq t < \infty \] (1.3)

and

\[ x'' + a(t)x = g(t,x,x'), \quad 0 \leq t < \infty \] (1.4)
where \( a(t) \in C(0, \infty) \) and \( g(t,x,x') \) is continuous on \([0, \infty) \times R \times R\). From some theorems of stability theory, Bownds [1] showed that (1.3) has a solution \( y(t) \) with property
\[
\limsup_{t \to \infty} (|y(t)| + |y'(t)|) > 0 \tag{1.5}
\]
He also established that (1.4) has the property (1.5) provided that the zero solution of (1.3) is stable and there exists a function \( \gamma(t) \in L(0, \infty) \) such that
\[
|g(t,x,x')| \leq \gamma(t) \left(|x| + |x'|\right)
\]
for \((t,x,x') \in [0, \infty) \times R \times R\).

Thus in the following section we shall extend the above results to systems (1.1) and (1.2). In section 3 we shall consider a nonhomogeneous system
\[
\frac{dX(t)}{dt} = A(t)X(t) + B(t), \ 0 \leq t < \infty \tag{1.6}
\]
where \( B(t) \) is a continuous vector for \( 0 \leq t < \infty \). We shall prove that there always exists a solution \( X(t) \) of (1.6) for which \( \limsup_{t \to \infty} ||X(t)|| > 0(= \infty) \), if there exists a solution \( Y(t) \) of (1.1) for which \( \limsup_{t \to \infty} ||Y(t)|| > 0(= \infty) \). Here \( || \cdot || \) is an appropriate vector (or matrix) norm.

2. ASYMPTOTIC BEHAVIOR FOR (1.1) AND (1.2).

Before stating main theorems, let us recall a theorem from Coppel [2, p. 60].

THEOREM 2.1. (Hartman [2, p. 60]). Suppose that, for every solution \( Y(t) \) of (1.1), the limit
\[
\lim_{t \to \infty} ||Y(t)|| \tag{2.1}
\]
exists and is finite. If there exists a nontrivial solution \( Y(t) \) of (1.1) for which the limit (2.1) is zero, then
\[
\int_{t_0}^{t} t_A(s)ds \to -\infty \ \text{as} \ t \to \infty.
\]

From the above theorem we will obtain the following corollary which is a generalization of Theorem 1 in [1].

COROLLARY 2.1. Suppose that
\[
\int_{t_0}^{\infty} t_A(s)ds < \infty.
\]
Then there exists a nontrivial solution \( Y(t) \) of (1.1) for which
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\[
\lim_{t \to \infty} \|Y(t)\| > 0.
\]

PROOF. Suppose, to the contrary, that all solutions \(Y(t)\) of (1.1) satisfy

\[
\lim_{t \to \infty} \|Y(t)\| = 0.
\]

From Theorem 2.1 we obtain

\[
\int_{t_0}^{t} t A(s)ds \to -\infty \text{ as } t \to \infty.
\]

This leads to a contradiction. The corollary then follows.

Throughout this paper we shall denote \(\Phi(t)\), the fundamental matrix of (1.1) with initial condition \(\Phi(0) = I\) (identity matrix).

Now we shall prove the following theorem via the Schauder-Tychonoff Theorem [2, p. 9].

THEOREM 2.2. Suppose that the null solution of (1.1) is stable and that there exists a solution \(Y(t)\) of (1.1) for which

\[
\lim_{t \to \infty} \|Y(t)\| > 0. \tag{2.2}
\]

Suppose also that there exists \(\gamma(t) \in L^1_{[t_0, \infty)}\) such that for some positive constant \(\ell\)

\[
\|F(t,x)\| \leq \gamma(t)\|x\|^{\ell}. \tag{2.3}
\]

Then there exists a nontrivial solution \(X(t)\) of (1.2) for which

\[
\lim_{t \to \infty} \|X(t)\| > 0.
\]

PROOF. Since the null solution of (1.1) is stable, there exists a positive constant \(k\) such that

\[
\|\Phi(t) \Phi^{-1}(s)\| \leq k \tag{2.4}
\]

for all \(0 \leq t \leq s\) and there exists a nontrivial solution \(Y(t)\) of (1.1) for which (2.2) holds and

\[
\|Y(t)\| \leq 1 - \varepsilon \tag{2.5}
\]

for \(t \geq t_0\) and for given small positive constant \(\varepsilon\) (<1). Since \(\gamma(t) \in L^1_{[t_0, \infty)}\), there exists \(T_0 (> t_0)\) such that

\[
k \int_{t}^{\infty} \gamma(s)ds < \varepsilon \quad \text{for all } t \geq T_0. \tag{2.6}
\]

Via the Schauder-Tychonoff Theorem we shall establish the existence of a solution.
of the integral equation
\[ X(t) = Y(t) - \Phi(t) \int_t^\infty \Phi(t) f(s, X(s))ds, \quad t \geq T_o. \] (2.7)

Consider the set
\[ F = \{ U; \ U(t) = X(t) \text{ is continuous on } J_o = [T_o, \infty) \text{ and } \| U(t) \| \leq 1 \text{ for } t \geq T_o \} \]
and define the operator \( T \) by
\[ TU(t) = Y(t) - \int_t^\infty \Phi(t) \Phi^{-1}(s) f(s, U(s))ds. \] (2.8)

First, we shall show that \( TF \subset F \). Taking the norm to both sides of (2.8) and using (2.3), (2.4), (2.5), and (2.6), we obtain for \( U \in F \)
\[
\|TU(t)\| \leq \|Y(t)\| + \int_t^\infty \|\Phi(t) \Phi^{-1}(s) f(s, U(s))\|ds
\leq 1 - \varepsilon + k \int_t^\infty \|f(s, U(s))\|ds
\leq 1 - \varepsilon + k \int_t^\infty \gamma(s) \|U(s)\|^\delta ds
\leq 1 - \varepsilon + k \int_t^\infty \gamma(s)ds
\leq 1 - \varepsilon + \varepsilon = 1.
\]
It is clear that \( TU(t) \) is continuous on \( J_o \). This proves \( TF \subset F \).

Second, we shall show that \( t \) is continuous. Suppose that the sequence \( \{U_n\} \) in \( F \) converges uniformly to \( U \) in \( F \) on every compact subinterval of \( J_o \). We claim that \( TU_n \) converges uniformly to \( TU \) on every compact subinterval of \( J_o \). Let \( \varepsilon_1 \) be a small positive number satisfying \( \varepsilon_1 < 1 \). Since \( \gamma(t) \in L_1[t_o, \infty) \), there exists \( T_1 > T_o \) so that for \( t \geq T_1 \)
\[
k \int_t^\infty \gamma(s)ds < \frac{\varepsilon_1}{4}. \] (2.9)
By the uniform convergence, there is an \( N = N(\varepsilon_1, T_1) \) such that if \( n \geq N \), then
\[
\|f(s, U_n(s)) - f(s, U(s))\| < \frac{\varepsilon_1}{2kT_1}, \quad T_o \leq s \leq T_1.
\] (2.10)
Then using (2.8), (2.9), (2.10), (2.3), (2.4), and the fact that \( \|U_n(t)\| \leq 1 \) and \( \|U(t)\| \leq 1 \) for \( T_o \leq t < \infty \), we obtain the following inequalities
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\[ |T_{U_{n}}(t) - T_{U}(t)| = |\int_{t}^{T} \Phi(t)^{-1}(s)f(s, U_{n}(s))ds - \int_{T}^{\infty} \Phi(t)^{-1}(s)f(s, U(s))ds| \leq \int_{t}^{T} |\Phi(t)^{-1}(s)||f(s, U_{n}(s)) - f(s, U(s))|ds \]

\[ + \int_{T_{1}}^{\infty} |\Phi(t)^{-1}(s)||f(s, U_{n}(s))|ds + \int_{T_{1}}^{\infty} |\Phi(t)^{-1}(s)||f(s, U(s))|ds \leq k \int_{t}^{T} |f(s, U_{n}(s)) - f(s, U(s))|ds + 2k \int_{T_{1}}^{\infty} \gamma(s)ds \leq \frac{\varepsilon_{1}}{2} + \frac{\varepsilon_{1}}{2} = \varepsilon_{1} \quad \text{for } n \geq N. \]

This shows that \( T_{U_{n}} \) converges uniformly to \( T_{U} \) on every compact subinterval of \( J_{0} \). Hence \( T \) is continuous.

Third, we claim that the functions in the image set \( T_{F} \) are equicontinuous and bounded at every point of \( J_{0} \). Since \( T_{F} \subset F \), it is clear that the functions in \( T_{F} \) are uniformly bounded. Now we show that they are equicontinuous at each point of \( J_{0} \). For each \( U \in F \), the function \( z(t) = T_{U}(t) \) is a solution of the linear system

\[ \frac{dv}{dt} = A(t)v + f(t, U(t)) \]

Since \( |z(t)| = |T_{U}(t)| \leq 1 \) and \( |f(t, U(t))| \) is uniformly bounded for \( U \in F \) on any finite \( t \) interval, we see that \( \frac{dz}{dt} \) is uniformly bounded on any finite interval. Therefore, the set of all such \( z \) is equicontinuous at each point of \( J_{0} \) (see [2, p.6]).

All of the hypotheses of the Schauder-Tychonoff Theorem are satisfied. Thus there exists a \( U \in F \) such that \( U(t) = T_{U}(t) \); that is, there exists a solution \( X(t) \) of

\[ X(t) = Y(t) - \Phi(t) \int_{t}^{\infty} \Phi^{-1}(s) f(s, x(s))ds \]

Thus, from the hypotheses and the above equality, we obtain

\[ \lim_{t \to \infty} \sup_{t} ||X(t) - Y(t)|| = 0 \quad (2.11) \]

Since \( \limsup_{t \to \infty} ||Y(t)|| > 0 \), (2.11) implies that \( \limsup_{t \to \infty} ||X(t)|| > 0 \). This proves the theorem.

It is clear that (1.4) can be written as the form (1.2) with
where \( X = \text{colum}(x,x') \). Thus we can apply Theorem 2.2 to (1.4) to obtain the following corollary which is a generalization of Theorem 2 in [1].

**COROLLARY 2.2.** Suppose that the null solution of (1.3) is stable and that there exists \( y(t) \in L^1_{[t_0,\infty)} \) such that for some positive constant \( \ell \)

\[ ||g(t,x,x')|| \leq y(t) (|x| + |x'|)^2. \]

Then there exists a nontrivial solution \( x(t) \) of (1.4) for which

\[ \limsup_{t \to \infty} (|x| + |x'|) > 0. \]

**PROOF.** Since \( t_r A(t) = 0 \) for Corollary 2.1, we know that there exists a solution \( Y(t) \) of (1.1) for which

\[ \limsup_{t \to \infty} ||Y(t)|| > 0. \]

If we take \( ||X|| = |x| + |x'| \), then the corollary follows from Theorem 2.2.

3. ASYMPTOTIC BEHAVIOR FOR (1.6).

In this section we shall show that if there exists a solution \( Y(t) \) of (1.1) for which \( \limsup_{t \to \infty} ||Y(t)|| > 0 (= \infty) \), then there exists a solution \( X(t) \) of (1.6) for which \( \limsup_{t \to \infty} ||X(t)|| > 0 (= \infty) \).

**THEOREM 3.1.** Suppose that there exists a solution \( Y(t) \) of (1.1) for which

\[ \limsup_{t \to \infty} ||Y(t)|| > 0. \]  \[ (3.1) \]

Then there exists a solution \( X(t) \) of (1.6) for which

\[ \limsup_{t \to \infty} ||X(t)|| > 0. \]

**PROOF.** From the variation of constants formula we know that any solution \( X(t) \) of (1.6) can be written as the form below

\[ X(t) = \phi(t)c + \phi(t) \int_0^t \phi^{-1}(s) B(s)ds \]  \[ (3.3) \]

Hence we shall choose \( c \) so that \( Y(t) = \phi(t)c \) satisfies (3.1).

First, let us suppose

\[ \limsup_{t \to \infty} ||\phi(t) \int_0^t \phi^{-1}(s) B(s)ds|| > 0. \]  \[ (3.4) \]
Let $X_1(t) = X(t) - Y(t)$. It is clear that $X_1(t)$ is a solution of (1.6). Thus from (3.3) and (3.4) we obtain
\[
\limsup_{t \to \infty} ||X_1(t)|| = \limsup_{t \to \infty} ||X(t) - Y(t)|| \\
= \limsup_{t \to \infty} ||\phi(t) \int_0^t \phi^{-1}(s) B(s) ds|| > 0.
\]
Thus there exists a solution $X_1(t)$ of (1.6) for which (3.2) holds.

Second, suppose that
\[
\lim_{t \to \infty} \phi(t) \int_0^t \phi^{-1}(s) B(s) ds = 0.
\]
Taking the norm to both sides of (3.3) and using (3.1) and (3.5) we obtain
\[
\limsup_{t \to \infty} ||X(t)|| \geq \limsup_{t \to \infty} (||Y(t)|| - ||\phi(t) \int_0^t \phi^{-1}(s) B(s) ds||) \\
\geq \limsup_{t \to \infty} ||Y(t)|| - \limsup_{t \to \infty} ||\phi(t) \int_0^t \phi^{-1}(s) B(s) ds|| \\
\geq \limsup_{t \to \infty} ||Y(t)|| > 0.
\]
This shows that $X(t)$ satisfies (3.2). The theorem then follows.

Using the same argument as Theorem 3.1 we also can obtain the following theorem.

THEOREM 3.2. Suppose that there exists a solution $Y(t)$ of (1.1) for which
\[
\limsup_{t \to \infty} ||Y(t)|| = \infty.
\]
Then there exists a solution $X(t)$ of (1.6) for which
\[
\limsup_{t \to \infty} ||X(t)|| = \infty.
\]

PROOF. Since the proof is almost the same as Theorem 3.1, we shall omit the detail.

REMARKS. It is interesting to note that Hatvani and Pintér [3] have studied this type of problem for equation (1.4).

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