ON BELLMAN-BIHARI INTEGRAL INEQUALITIES

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(Received October 15, 1980)

ABSTRACT. Integral inequalities of the Bellman-Bihari type are established for integrals involving an arbitrary number of independent variables.

KEY WORDS AND PHRASES. Integral inequalities, differential inequalities.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 34A40, 35B45.

1. INTRODUCTION.

In a number of recent papers, Dhongade and Deo [1] and Pachpatte [2,3,4] have generalized the well known Bellman inequality [5] and Bihari's generalization of it [6] in several different directions. Although the results concern only functions of a single variable, it was shown in [7] that corresponding inequalities also hold for functions of several independent variables. The purpose of this note is to show that the technique employed in [7] can be profitably utilized to establish more general integral inequalities of the Bellman-Bihari type in any number of independent variables. We present here some of the results along this line.

As in [7] we assume that all the functions under discussion are defined in a bounded domain $R$ of $\mathbb{R}^n$ which, for convenience, is assumed to contain the origin. The symbol $x < y$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are any two points of $R$, means $x_i < y_i$ for $i = 1, \ldots, n$. We also adopt the notation

$$\int_0^X f(s)ds = \int_0^{X_n} \cdots \int_0^{X_1} f(s_1, \ldots, s_n)ds_1 \cdots ds_n$$
2. **MAIN RESULTS.**

Our first result is a variation of Theorem 3 of [7].

**THEOREM 1.** Let \( u, f, \) and \( g \) be continuous and nonnegative in \( \mathbb{R} \) and let \( a \) be continuous, positive and nondecreasing in \( \mathbb{R} \). Let \( W: [0, \infty) \to [0, \infty) \) be continuously differentiable and nondecreasing such that

\[
-\frac{1}{u} W(u) \leq W(-u), \quad u > 0, \quad v > 0
\]  

(2.1)

Then the inequality

\[
u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)W(u)dt]ds \]  

(2.2)

implies

\[
u(x) \leq a(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds] \]  

if \( g(x) \leq f(x) \) or

\[
u(x) \leq a(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s g(t)dt)ds] \]  

(2.3)

if \( f(x) \leq g(x) \), where \( G^{-1} \) is the inverse of the function

\[
G(w) = \int_{w_0}^w \frac{dr}{r+W(r)}, \quad w > w_0 > 0
\]  

(2.5)

provided \( G(1) + \int_0^x f(t)dt \) lies in the domain of \( G^{-1} \).

**PROOF.** Since \( a > 0, \ W > 0 \) and both are nondecreasing, and by (2.1), we may rewrite (2.2) in the form

\[
u(x) \leq 1 + \int_0^x f(s)[m(s) + \int_0^s g(t)W(m)dt]ds \]  

where \( m(x) \leq u(x)/a(x) \). If we set \( v(x) \) equal to the right hand side of (2.6) and differentiate, we find

\[
D_1 \ldots D_n v(x) = f(x)m(x) + \int_0^x g(t)W(m)dt \leq f(x)v(x) + \int_0^x g(t)W(v)dt
\]  

(2.7)

where \( D_i \) indicates differentiation with respect to \( x_i, \ i = 1, \ldots, n \).
Let us define
\[ w(x) = v(x) + \int_0^x g(t) W(v) \, dt \]  
(2.8)
and assume \( g(x) \leq f(x) \). Then, by differentiating (2.8) and using (2.7), we obtain
\[ D_1 \ldots D_n w(x) = D_1 \ldots D_n v(x) + g(x) W(v) \]
(2.9)
\[ \leq f(x) w(x) + g(x) W(w) \]
\[ \leq f(x) (w(x) + W(w)) \]

Set \( S(x) = w(x) + W(w) \). Following the technique in [7], we observe from (2.9) that
\[ \frac{S(x) D_1 \ldots D_n w(x)}{S(x)^2} \leq \frac{D_1 S(x) D_2 \ldots D_n w(x)}{S(x)^2} \]
or
\[ D_1 \left( \frac{D_2 \ldots D_n w(x)}{S(x)} \right) \leq f(x) \]
Note that, from the hypotheses, it follows that \( D_i (w(x) + W(w)) > 0 \), for \( i = 1, 2, \ldots, n \). Hence, integrating with respect to \( x_1 \) from 0 to \( x_1 \), we find
\[ \frac{D_2 \ldots D_n w(x)}{S(x)} \leq \int_0^{x_1} f(s_1, x_2, \ldots, x_n) \, ds_1 \]  
(2.10)
Similarly, since
\[ \frac{D_2 S(x) (D_3 \ldots D_n w(x))}{S(x)^2} \geq 0 \]
the left hand side of (2.10) can be replaced by
\[ D_2 \left( \frac{D_3 \ldots D_n w(x)}{S(x)} \right) \leq \int_0^{x_1} f(s_1, x_2, \ldots, x_n) \, ds_1 \]
By integrating this from 0 to \( x_2 \), we obtain
\[ \frac{D_3 \ldots D_n w(x)}{S(x)} \leq \int_0^{x_2} \int_0^{x_1} f(s_1, s_2, x_3, \ldots, x_n) \, ds_1 \, ds_2 \]
Continuing in this manner, we have after \( (n-1) \) steps
\[ \frac{D_n w(x)}{S(x)} \leq \int_0^{x_{n-1}} \ldots \int_0^{x_1} f(s_1^{'}, \ldots, s_{n-1}, x_n) \, ds_1^{'} \ldots ds_{n-1}^{'} \]  
(2.11)
With the function $G(w)$ defined in (2.5), we note that
\[ D_n G(w) = G'(w) D_n w(x) = D_n w(x)/(w(x) + W(w)). \]
Hence, integration of (2.11) from 0 to $x_n$ yields
\[ G(w(x_1, \ldots, x_n)) - G(w(x_1, \ldots, x_{n-1}, 0)) \leq \int_0^x f(s) \, ds \]
or
\[ w(x) \leq G^{-1}(G(1) + \int_0^x f(s) \, ds) \quad (2.12) \]
since $w(x) = v(x) = 1$ when $x_1 = 0$ for any $i$, $1 \leq i \leq n$.

From (2.7) and (2.8) we have
\[ D_1 \ldots D_n v(x) \leq f(x) w(x) \quad (2.13) \]
Substituting for $w(x)$ from (2.12) and integrating (2.13), we finally obtain
\[ v(x) \leq 1 + \int_0^x f(s) G^{-1}(G(1) + \int_0^s f(t) \, dt) \, ds \quad (2.14) \]
The inequality (2.3) follows from (2.6), (2.14), and the fact that $m(x) = u(x)/a(x)$.

If $f(x) < g(x)$, then we need only replace $f$ by $g$ in the last line of (2.9) to obtain again (2.12) with $f$ replaced by $g$. The result (2.4) then follows in the same fashion.

Our next theorem combines the feature of Theorems 1 and 2 of [7].

THEOREM 2. Let $u$, $f$, $g$, and $h$ be continuous and nonnegative functions in $\mathbb{R}$, and let $a$ be continuous, positive, and nondecreasing in $\mathbb{R}$. Let $Z: [0, \infty) \to [0, \infty)$ satisfy the same conditions as $W$ in Theorem 1 such that $Z$ is submultiplicative.

If $u$ satisfies
\[ u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)u(t) \, dt] \, ds + \int_0^x h(s)Z(u) \, ds \quad (2.15) \]
then
\[ u(x) \leq a(x) p(x) H^{-1}(H(1) + \int_0^x h(s)Z(p) \, ds) \quad (2.16) \]
where
\[ p(x) = 1 + \int_0^x f(s) \exp \left( \int_0^s (f(t) + g(t)) \, dt \right) \, ds \quad (2.17) \]
and $H^{-1}$ is the inverse of the function.
The proof of this theorem makes use of the following result which we state as a lemma. This was established in [7] as Theorem 1.

LEMMA. Under the hypotheses of Theorem 2, the inequality

$$u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)u(t)dt]ds$$

implies

$$u(x) \leq a(x)[1 + \int_0^x f(s)\exp \int_0^s (f(t) + g(t))dt]ds].$$

PROOF of Theorem 2. As in Theorem 1 we rewrite (2.15) in the form

$$m(x) \leq 1 + \int_0^x f(s)[m(s) + \int_0^s g(t)m(t)dt]ds$$

$$+ \int_0^x h(s)Z(m)ds$$

(2.19)

If we set

$$v(x) = 1 + \int_0^x h(s)Z(m)ds$$

(2.20)

then (2.19) becomes

$$m(x) \leq v(x) + \int_0^x f(s)[m(s) + \int_0^s g(t)m(t)dt]ds.$$

Hence, by the lemma, we have

$$m(x) \leq v(x)(1 + \int_0^x f(s)\exp \int_0^s (f(t) + g(t))dt]ds)$$

$$\leq v(x)p(x)$$

(2.21)

Since $Z$ is submultiplicative, we note that $Z(m) \leq Z(v)Z(p)$. Therefore, differentiating (2.20) with respect to $x_1, \ldots, x_n$, we find

$$D_1 \ldots D_n v(x) = h(x)Z(m)$$

$$\leq h(x)Z(v)Z(p)$$
or
\[ \frac{D_1 \cdots D_nv(x)}{Z(v)} \leq h(x)Z(p) \] (2.22)

By the same argument as in the proof of Theorem 1, we can integrate (2.22) to obtain
\[ H(v(x_1, \ldots, x_n)) - H(v(x_1, \ldots, x_{n-1}, 0)) \leq \int_0^x h(s)Z(p)ds \]
where \( H(v) \) is defined by (2.18). This gives
\[ v(x) \leq H^{-1}(H(1) + h(s)Z(p)ds) \] (2.23)

The substitution of (2.23) in (2.21) yields the inequality (2.16) since
\[ m(x) = u(x)/a(x) \] when \( g(x) = 0 \), Theorem 2 reduces to Theorem 3 of [7].

By combining Theorems 1 and 2, we finally have

**THEOREM 3.** Let \( u, a, f, g, h, \) and \( Z \) be as in Theorem 2 and let \( W \) be as in Theorem 1. If \( u \) satisfies
\[ u(x) \leq a(x) + f(s)[u(s) + g(t)W(m)dt]ds \] (2.24)
\[ + \int_0^x h(s)Z(u)ds, \] where \( g(x) \leq f(x) \) then
\[ u(x) \leq a(x)q(x)H^{-1}(H(1) + \int_0^x h(s)Z(q)ds) \] (2.25)

where
\[ q(x) = 1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds \] (2.26)

\( G^{-1} \) is the inverse of the function defined in (2.5) and \( H^{-1} \) is the inverse of the function defined in (2.18).

**PROOF.** We rewrite (2.24) in the form
\[ m(x) \leq v(x) + \int_0^x f(s)[m(s) + g(t)W(m)dt]ds \] (2.27)
where
\[ v(x) = 1 + \int_0^x h(s)Z(m)ds \] (2.28)
with \( m(x) = u(x)/a(x) \). Then according to Theorem 1, we have

\[
 m(x) \leq v(x)\left[1 + \int_0^X f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds\right] \leq v(x)q(x) \tag{2.29}
\]

Since \( Z(m) \leq Z(v)Z(q) \), we obtain from (2.28)

\[
 D_1\ldots D_n v(x) = h(x)Z(m) \leq h(x)Z(v)Z(q)
\]

With \( H(v) \) defined by (2.18), we obtain as in the proof of Theorem 2

\[
 v(x) \leq H^{-1}(H(1) + \int_0^X h(s)Z(q)ds)
\]

The substitution of this for \( v(x) \) in (2.29) leads to the desired inequality (2.25).

Observe that, when \( h(x) = 0 \), (2.25) reduces to (2.3); when \( W = u \), it agrees with (2.16) with \( g \) replaced by \( f \) in view of the condition \( g \leq f \).

We remark that our Theorems 1, 2, and 3 correspond respectively to Theorems 4, 2, and 5 of [4]. From the argument presented above, we readily see that other more general integral inequalities can also be established for \( n \) independent variables along the lines considered in [1] and [4].

REFERENCES


