ABSTRACT. An improved technique is presented for the stability analysis of Robe's 3-body problem which gives more accurate results for the transition curves in the parameter plane than does Robe's paper.

A novel property of the system of differential equations describing the motion is used, which reduces the computer time by more than 50%.

KEY WORDS AND PHRASES. 3-body problem, stability, transition curve, Floquet-theory.

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1. INTRODUCTION.

In a recent paper Robe [1] presented a new kind of restricted three body problem, where one body \( m_1 \) is a rigid spherical shell, filled with an homogeneous incompressible fluid of density \( \rho_1 \), where a second body \( m_2 \) is a mass point outside the shell, and where \( m_3 \) is a small solid sphere of density \( \rho_3 \), restricted to move inside the shell, its motion determined by the attraction of \( m_2 \) and the buoyancy force due to the fluid \( \rho_1 \).

There exists a solution with \( m_3 \) at the center of the shell while \( m_2 \) describes a Keplerian orbit around it. Robe investigated the stability of this configuration under the assumption that the mass of \( m_3 \) is infinitesimal. The linearized equations
of motion in the neighborhood of this equilibrium are

\[ \ddot{x} - 2\dot{y} = \left\{ \frac{1 + 2\mu}{1 + \cos v} - \frac{K(1 - e^2)^3}{(1 + \cos v)^4} \right\} x \] (1.1)

\[ \ddot{y} + 2\dot{x} = \left\{ \frac{1 - \mu}{1 + \cos v} - \frac{K(1 - e^2)^3}{(1 + \cos v)^4} \right\} y \] (1.2)

\[ \ddot{z} + z = \left\{ \frac{1 - \mu}{1 + \cos v} - \frac{K(1 - e^2)^3}{(1 + \cos v)^4} \right\} z \] (1.3)

where

\[ K = \frac{4\pi}{3} \cdot \frac{\mu_1 a^3}{(m_1 + m_2)} \left( 1 - \frac{\mu_1}{\mu_3} \right) \]

\[ \mu = \frac{m_2}{(m_1 + m_2)} \]

\[ a = \text{semi-major axis of the Keplerian orbit.} \]

These equations are referred to a coordinate system Oxyz, where 0 is the center of \( m_1 \), Ox points to \( m_2 \) and Oxy is the plane of the Keplerian orbit.

If \( \rho_1 = 0 \) (shell empty) or \( \rho_1 = \rho_3 \), then \( K = 0 \) and the equations of motion become

\[ \ddot{x} - 2\dot{y} = (1 + 2\mu) r \cdot x \] (1.4)

\[ \ddot{y} + 2\dot{x} = (1 - \mu) r \cdot y \] (1.5)

\[ \ddot{z} + z = (1 - \mu) r \cdot z \] (1.6)

with

\[ r = \frac{1}{1 + \cos v} \cdot \]

Equations (1.4) and (1.5) describe the motion in the orbital plane. Robe investigated the stability in the orbital plane by means of the Floquet-theory.

However, one can separate the fourth-order system (1.4), (1.5) into two independent second-order systems.

2. THE TRANSFORMATION TO SECOND-ORDER SYSTEMS.

Using \( \xi = [x \ y] \) and \( \eta = [\dot{x} \ \dot{y}] \),

equations (1.4) and (1.5) can be written
STABILITY ANALYSIS IN ROBE'S THREE BODY PROBLEM

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
0 & E \\
rC_0 & 2D
\end{bmatrix}
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
\] (2.1)

where
\[
E = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix},
\]

and
\[
C_0 = \begin{bmatrix}
1 + 2\mu & 0 \\
0 & 1 - \mu
\end{bmatrix}
\]

Now we make the following transformation (Tschauner [2])
\[
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix} = \begin{bmatrix}
E & E \\
P_1 & P_2
\end{bmatrix}
\begin{bmatrix}
\delta \\
\epsilon
\end{bmatrix}
\] (2.2)

and obtain
\[
\begin{bmatrix}
\dot{\delta} \\
\dot{\epsilon}
\end{bmatrix} = \frac{1}{P_2 - P_1} \cdot \begin{bmatrix}
P_2P_1 - rC_0 - 2DP_1 + P_1' & P_2^2 - rC_0 - 2DP_2 + P_2' \\
-P_2^2 + rC_0 + 2DP_1 - P_1' & -P_2P_1 + rC_0 + 2DP_2 - P_2'
\end{bmatrix}
\begin{bmatrix}
\delta \\
\epsilon
\end{bmatrix}
\]

Making the nondiagonal elements zero, we obtain
\[
\begin{bmatrix}
\dot{\delta} \\
\dot{\epsilon}
\end{bmatrix} = \begin{bmatrix}
P_1 & 0 \\
P_2 & 0
\end{bmatrix}
\begin{bmatrix}
\delta \\
\epsilon
\end{bmatrix}
\] (2.3)

where \(P_1\) and \(P_2\) are two different solutions of the Riccati equation
\[
\dot{P} = 2DP - P^2 + rC_0.
\] (2.4)

Using
\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix},
\]
equation (2.4) becomes

\[
\begin{align*}
\dot{p}_{11} &= 2p_{21} - p_{11}^2 - p_{12}P_{21} + r(1 + 2u) \\
\dot{p}_{22} &= -2p_{12} - p_{22}^2 - p_{12}P_{21} + r(1 - u) \\
\dot{p}_{12} &= 2p_{22} - p_{12}(p_{11} + p_{22}) \\
\dot{p}_{21} &= -2p_{11} - p_{21}(p_{11} + p_{22})
\end{align*}
\]  

(2.5)

Now let

\[
\begin{align*}
p_{11} &= w + z \\
p_{22} &= w - z \\
p_{12} &= u - v + 1 \\
p_{21} &= u + v - 1
\end{align*}
\]  

(2.6)

Then equation (2.5) becomes

\[
\begin{align*}
\dot{w} &= -1 - w^2 - z^2 - u^2 + v^2 + r(\frac{2 + u}{2}) \\
\dot{z} &= 2u - 2wz + r \frac{3}{2} u \\
\dot{u} &= -2z - 2uw \\
\dot{v} &= -2vw
\end{align*}
\]  

(2.7)

The last two equations yield

\[
\begin{align*}
w &= \frac{v}{2} \left( \frac{1}{v} \right) \\
z &= -\frac{v}{2} \left( \frac{u}{v} \right)
\end{align*}
\]

(2.8)

If we use \( p = \frac{1}{v} \) and \( q = \frac{u}{v} \), we get

\[
\begin{align*}
w &= \frac{p}{2p} \\
z &= -\frac{q}{2p}
\end{align*}
\]
Substituting (2.8) into the first two equations of (2.7), we have
\[ \ddot{p} - \frac{1}{2} \dot{p}^2 + [2 - r(2 + \mu)] p^2 - 2 = -2(q^2 + \frac{1}{4} \dot{r}^2) \quad (2.9) \]
\[ \ddot{q} + 4q = -3rp\mu \quad (2.10) \]

Now if we let
\[ rp = k_0 + k_1 \cos v, \quad (2.11) \]
we find as a solution for (2.10)
\[ q = -\frac{3}{4} \mu k_0 - \mu k_1 \cos v + k_2 \cos^2 v \quad (2.12) \]

If we substitute the solutions (2.11) and (2.12) into (2.9), we obtain (by identification) the values for \( k_0, k_1 \) and \( k_2 \) as functions of \( \mu \) and \( e \).

For \( k_0 \), we obtain
\[ k_0 = \pm \frac{4}{c} \]

with
\[ c = \sqrt{\frac{4e^4}{(3\mu + 1)^2} + \frac{4e^2\mu}{(3\mu + 1)^2} - (\mu + 3) + \mu(9\mu - 8)} \quad (2.13) \]

This will yield two different solutions, \( P_1 \) and \( P_2 \), if \( c \neq 0 \) and \( c^2 > 0 \).

So \( c(\mu, e) = 0 \) will give us (analytically) a transition curve in the \( \mu-e \) plane, which corresponds to one of the transition curves (IE) that Robe obtained numerically.

This curve can be written as
\[ \frac{4e^4}{(3\mu + 1)^2} + \frac{4e^2\mu}{(3\mu + 1)^2} (\mu + 3) + \mu(9\mu - 8) = 0 \quad (2.14) \]

The elements \( p_{ij} \) of \( P_1 \) and \( P_2 \) are
\[ p_{11} = \frac{-(2\mu + 1)e \sin v}{(3\mu + 1 + e\cos v)(1 + e\cos v)} \quad (2.15) \]
3. STABILITY ANALYSIS.

Now we perform a stability analysis using the Floquet theory, on the two independent second order systems (2.3)

\[ \dot{\delta} = P_1 \delta \]  
\[ \dot{\epsilon} = P_2 \epsilon \]  

Both equations admit solutions for which

\[ u(v + 2\pi) = s_i u(v) \quad (i = 1, 2) \]

where \( s_i \) are the roots of the characteristic equation

\[ \det [X^{-1}(v)X(v + 2\pi) - sE] = 0 \]  

where \( X(v) \) is a fundamental solution matrix of (3.1) (or (3.2)). Equation (3.3) can be written

\[ s^2 - 2\alpha s + 1 = 0. \]  

For stable solutions,

\[ |\alpha| < 1 \]  

Thus the transition curves in the \((\mu-e)\) plane, separating stable and non-stable regions, will be given by

\[ |\alpha| = 1 \]

or

\[ s = \pm 1 \]  

Taking \( X(v = 0) = E \), equation (3.3) becomes

\[ \det [X(2\pi) - sE] = 0 \]

where \( X(2\pi) = \begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix} \) is the monodromy-matrix and \( \alpha^2 - \beta \gamma = 1. \)

It can be shown (R. Meire and A. Vanderbauwhede [3]) that
where

\[ S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

This implies that we only have to integrate the equations over \( \pi \) instead of \( 2\pi \).

One can also prove that for \( s = \pm 1 \), one of the elements of \( X(\pi) \) becomes zero which saves an additional 10% of computer time.

4. RESULTS.

We applied this method to the equations of (3.1) and (3.2). Equation (3.1) yielded two transition curves: FM and FE. Equation (3.2) yielded one transition curve HM. All results are given in Fig. 1.

The curve IE is the analytically obtained curve from equation (2.14).

Along

\[
\begin{align*}
\text{FM} & : \beta = 0 \text{ and } \alpha = -1 \\
\text{FE} & : \gamma = 0 \text{ and } \alpha = -1 \\
\text{HM} & : \beta = \gamma = 0 \text{ and } \alpha = 1
\end{align*}
\]

The intersection of the curves FE and IE can be obtained very accurately (Robe was not able to give precise coordinates).

The point E is determined as that point on the curve IE for which the characteristic roots of (3.1) are -1. The coordinates of the interesting points in the \( \mu - e \) plane are

<table>
<thead>
<tr>
<th>Point</th>
<th>( \mu )</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>0.928053, ( \frac{5 + \sqrt{97}}{16} )</td>
<td>0</td>
</tr>
<tr>
<td>I</td>
<td>0.8888, ( \frac{8}{9} )</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0.8596848</td>
<td>0.4531741</td>
</tr>
</tbody>
</table>

The stable region consists of the shaded area in Fig. 1 and is now determined much
more accurately than in Robe's paper where the fourth-order system (2.1) was used.

Fig. 1: Stability-regions in the μ-e-plane

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