THE CONVOLUTION-INDUCED TOPOLOGY ON L_\infty(G) AND LINEARLY DEPENDENT TRANSLATES IN L_1(G)

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ABSTRACT. Given a locally compact Hausdorff group G, we consider on L_\infty(G) the \tau_c-topology, i.e. the weak topology under all convolution operators induced by functions in L_1(G). As a major result we characterize the trigonometric polynomials on a compact group as those functions in L_1(G) whose left translates are contained in a finite-dimensional set. From this, we deduce that \tau_c is different from the w*-topology on L_\infty(G) whenever G is infinite. As another result, we show that \tau_c coincides with the norm-topology if and only if G is discrete. The properties of \tau_c are then studied further and we pay attention to the \tau_c-almost periodic elements of L_\infty(G).

KEY WORDS AND PHRASES. Locally compact group, convolution operator, topology induced by convolution, linearly dependent translates, almost periodic functions.

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1. INTRODUCTION.

The reader intending to read the following paper should have some familiarity with such basic texts as Hewitt and Ross or Dunford and Schwartz.

For a locally compact Abelian group G, Argabright and Gil de Lamadrid [1] considered almost periodicity of measures with respect to several topologies. A special case of this general notion, namely almost periodicity with respect to the \tau_c-topology on L_\infty(G), has been used in Crombez and Govaerts [2] in order to characterize those multipliers from L_1(G) to L_\infty(G) which are almost periodic in the strong
operator topology. Throughout this paper, unless explicitly stated otherwise, $G$ will denote a locally compact Hausdorff group with left Haar measure. For such an arbitrary $G$ the $\tau_c$-topology is not weaker than the $w^\times$-topology and not stronger than the norm topology on $L_\infty(G)$. The question as to whether there are neighborhoods in the $\tau_c$-topology which are also neighborhoods in the $w^\times$-topology leads us to consider the apparently completely different problem of determining those functions $f \not= 0$ in $L_1(G)$ such that all left translates of $f$ are in a finite-dimensional subspace of $L_1(G)$ (a related problem was recently investigated in Edgar and Rosenblatt [3] for Abelian groups). We prove that such functions only exist for compact $G$, and then they are exactly the trigonometric polynomials. From this result we derive that the $\tau_c$-topology is always different from the $w^\times$-topology whenever $G$ is infinite. However, a further investigation shows that for compact $G$ these two topologies coincide on every norm-bounded subset of $L_\infty(G)$, and so we may conclude that for compact $G$ $L_1(G)$ is the dual of $(L_\infty(G), \tau_c)$. Among the other results we mention that except for discrete $G$ the $\tau_c$-topology is always different from the norm-topology (section 3), and that for fixed $g$ in $L_\infty(G)$ the map $s \mapsto g$ from $G$ to $(L_\infty(G), \tau_c)$ is continuous (section 4). In section 5 we give some further results about $\tau_c$-almost periodic functions.

For complex-valued functions $f$ and $g$ on $G$ and $a \in G$, we define the left translate $a^\circ f$ and the convolution $f * g$ by means of $a^\circ f(x) = f(ax)$ and $(f * g)(x) = \int_G f(xy)g(y^{-1})dy$ (we warn the reader that in some of the references, e.g. [4] and [5], different conventions are used). Each function $f$ in $L_1(G)$ induces by convolution an operator $T_f$ on $L_\infty(G)$; the weak topology on $L_\infty(G)$ under all convolution operators $T_f: L_\infty(G) \to (L_\infty(G), || ||_\infty)$ is denoted by $\tau_c$. By $w^\times$ and $|| ||_\infty$ we denote the $(L_\infty(G), L_1(G))$, i.e., weak $\times$ topology, and the essential supremum norm topology respectively, on $L_\infty(G)$. All other nonexplained notation is taken from Hewitt and Ross [6].

2. FUNCTIONS IN $L_1(G)$ WITH FINITE-DIMENSIONAL SPAN OF TRANSLATES.

From the definitions we immediately derive $w^\times \subseteq \tau_c \subseteq || ||_\infty$. Investigation of the possibility that some $\tau_c$-neighborhood is also a $w^\times$-neighborhood leads to a special class of functions in $L_1(G)$, as Proposition 1 shows. For convenience we take as a subbase at 0 for $w^\times$ the sets
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\( \{ h \in L^\infty(G) : \left| \int f(x)h(x^{-1})dx \right| < \epsilon \} \), where \( f \in L^1(G) \) and \( \epsilon > 0 \); we write \( \langle f, h \rangle \) for

\[ \int_G f(x)h(x^{-1})dx . \]

PROPOSITION 1. For \( 0 \neq f \in L^1(G) \) the following are equivalent:

(i) There exists an \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood determined by \( f \) and \( \epsilon \) is a \( \mathbb{W} \)-neighborhood.

(ii) The set of left translates of \( f \) is part of a finite-dimensional subspace of \( L^1(G) \).

(iii) There exist \( a_1, \ldots, a_n \) in \( G \) such that, for each \( a \in G \), scalars \( c_1, \ldots, c_n \) may be found such that

\[ \sum_{i=1}^n c_i a_i f \]

(iv) Given \( \epsilon > 0 \), there exists \( a_1, \ldots, a_n \) in \( G \) and \( \delta > 0 \) such that, for \( g \in L^\infty(G) \),

\[ \left| \sum_{i=1}^n c_i a_i f \right| < \epsilon \]

PROOF (i)\( \Rightarrow \) (ii). Suppose that the set \( \{ g \in L^\infty(G) : \left| \int f(x)g(x^{-1})dx \right| < \delta \} \) is a \( \mathbb{W} \)-neighborhood of zero. Then we may find functions \( f_i (i=1, \ldots, r) \) in \( L^1(G) \) and \( \delta > 0 \) such that, whenever \( g \in L^\infty(G) \) and \( \left| \int f_i(x)g(x^{-1})dx \right| < \delta \) for all \( i \), then \( \left| \int f_i g \right|_\infty < \epsilon \). Each \( f_i \) determines a linear functional on \( L^\infty(G) \); call \( N \) the intersection of their kernels. Since for any scalar \( c, cg \in N \) whenever \( g \in N \), there results that \( |c| \left| \int f g \right|_\infty < \epsilon \) for \( g \in N \) and for any scalar \( c \); hence \( f g = 0 \) for \( g \) in \( N \), or \( \int_G f(y)g(y^{-1})dy = 0 \) for any \( a \in G \) and \( g \) in \( N \). This means that, for given \( a \in G \), the linear functional determined by \( f \) may be written as a linear combination of the ones determined by the \( f_i (i=1, \ldots, r) \). So, given \( a \in G \), there exist scalars \( a_1, \ldots, a_r \) such that

\[ f = \sum_{i=1}^r a_i f_i \]

(ii)\( \Rightarrow \) (iii). Obvious. We may choose \( a_1, \ldots, a_n \) in \( G \) such that the set \( \{ a_i f \}_{i=1}^n \) is also a linearly independent set.

(iii)\( \Rightarrow \) (iv). We first remark that the assumption of (iii) implies that \( G \) is necessarily compact. Indeed, whenever (iii) is true the set \( \{ a : a \in G \} \) of left translates of \( f \) is a norm-bounded subset of a finite-dimensional subspace of \( L^1(G) \), and so this set is relatively compact with respect to the norm-topology of \( L^1(G) \). However, it was shown in Crombez and Govaerts [4] that for non-compact \( G \) only \( f = 0 \) has this property.

From this it also follows that there exists \( B > 0 \) such that for all \( a \in G \)

\[ \sum_{i=1}^n |c_i| < B \]
for the scalars figuring in (iii). Indeed, the function $a \mapsto f$ from $G$ to $L_1(G)$ is continuous, and its range is part of a finite-dimensional subspace $M$ of $L_1(G)$; assuming, as we may, that $\{a_i f\}_{i=1}^n$ is linearly independent, the function $$ f = \sum_{i=1}^n c_i a_i f(c_1, \ldots, c_n) $$ from $M$ to the $n$-dimensional complex space $\mathbb{C}^n$ is (well-defined and) linear, and hence continuous; so the composition of these two functions is continuous on the compact group $G$, from which the result follows.

Suppose then that (iii) is true, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $B_\delta < \varepsilon$, with $B$ as mentioned above. If $a_1, \ldots, a_n$ are as in (iii) and $| < a_i f, g > | < \delta$ for all $i = 1, \ldots, n$, then for $a \in G$ we have

$$ |(fg)(a)| = \left| \sum_{i=1}^n c_i a_i f(y)g(y^{-1})dy \right| < \sum_{i=1}^n |c_i| < < a_i f, g > < \varepsilon. $$

(iv)$\Rightarrow$(i). Obvious.

Statement (iii) in Proposition 1 leads to the following problem: determine those $f \in L_1(G)$ for which all left translates are contained in a finite-dimensional subset of $L_1(G)$. As remarked in the proof of the proposition such nonzero functions can exist only for compact $G$. To solve this problem, we use the theory of representations of compact groups as explained in Hewitt and Ross [6]. It is readily verified that the set of functions with the mentioned property is a linear subspace $\mathcal{V}$ of $L_1(G)$ containing all trigonometric polynomials. Proposition 2 shows that there are no other functions in $\mathcal{V}$. (For related results in the abelian case, we refer to Schwartz [7], and to the recent paper of Laird [8] and the references mentioned there.)

**Proposition 2.** Let $0 \neq f \in L_1(G)$ with $G$ compact. The set $\{a f : a \in G\}$ of left translates of $f$ is contained in a finite-dimensional space iff $f$ is a trigonometric polynomial on $G$.

**Proof.** We first remark that $f$ is a trigonometric polynomial iff the Fourier transform $\hat{f}$ of $f$ is such that $\hat{f}(\sigma) = 0$ except for a finite number of elements $\sigma$ in the dual object $\hat{G}$ of $G$ (see Hewitt and Ross [6], 28.39).

Let then $f \in L_1(G)$ be such that statement (iii) of Proposition 1 is true, i.e., $a = \sum_{i=1}^n c_i(a) a_i f$ (for fixed $n$) and $\sum_{i=1}^n |c_i(a)| \leq B$ (this was shown in the proof of (iii)$\Rightarrow$(iv) above). Taking the Fourier transform we obtain
Let $D$ be the set of those $\sigma \in \mathcal{L}$ for which $\hat{f}(\sigma)$ is different from zero. Then for each $\sigma \in D$ there is a subspace $M_\sigma \neq \{0\}$ in the representation space $H_\sigma$ of $U^{(\sigma)}$ such that $\overline{U}_a^{(\sigma)} = \sum_{i=1}^{n} c_i(a)\overline{U}_a^{(\sigma)} = 0$ on $M_\sigma$, $\forall a \in G$. We choose an element $\xi(\sigma)$ in $M_\sigma$ with $\|\xi(\sigma)\| = 1$. Since $U^{(\sigma)}$ is irreducible, the non-zero vector $\xi(\sigma)$ is a cyclic vector for $\overline{U}^{(\sigma)}$, which means that the set of all finite linear combinations of elements from $\{\overline{U}_a^{(\sigma)} \xi(\sigma) : a \in G\}$ is all of $H_\sigma$; but the set $\{\overline{U}_a^{(\sigma)} \xi(\sigma) : a \in G\}$ is spanned by the finitely many vectors $\overline{U}_a^{(\sigma)} \xi(\sigma), \ldots, \overline{U}_a^{(\sigma)} \xi(\sigma)$; hence, if $d_\sigma$ denotes the dimension of $H_\sigma$ we always have $d_\sigma \leq n$, for each $\sigma$ in $D$.

With the choice of $\xi(\sigma)$ we have $|\langle \overline{U}_a^{(\sigma)} \xi(\sigma), \xi(\sigma) \rangle| = 1$ for all $\sigma \in D$ and all $i \in \{1, \ldots, n\}$, where now $\langle, \rangle$ denotes the inner product on $H_\sigma$. If $D$ is infinite, we obtain an infinite family $\{\langle \overline{U}_a^{(\sigma)} \xi(\sigma), \xi(\sigma) \rangle, \ldots, \overline{U}_a^{(\sigma)} \xi(\sigma) \}_{\sigma \in D}$ in a compact set in the $n$-dimensional complex space, and so it has a cluster point; this means that, given $0 < \varepsilon < \frac{1}{n}$, there exist different $\sigma_1$ and $\sigma_2$ in $D$ such that $|\langle \overline{U}_a^{(\sigma_1)} \xi(\sigma_1), \xi(\sigma_1) \rangle - \langle \overline{U}_a^{(\sigma_2)} \xi(\sigma_2), \xi(\sigma_2) \rangle| \leq \varepsilon$ for all $i$. For each $a$ in $G$ we then have

\[
|\langle \overline{U}_a^{(\sigma_1)} \xi(\sigma_1), \xi(\sigma_1) \rangle - \langle \overline{U}_a^{(\sigma_2)} \xi(\sigma_2), \xi(\sigma_2) \rangle| = \left| \sum_{i=1}^{n} c_i(a) \langle \overline{U}_a^{(\sigma_1)} \xi(\sigma_1), \xi(\sigma_1) \rangle - \langle \overline{U}_a^{(\sigma_2)} \xi(\sigma_2), \xi(\sigma_2) \rangle \right| \leq \varepsilon
\]

Assuming that the Haar measure of the compact group $G$ is normalised, it follows that

\[
\left| \int_G \langle \overline{U}_a^{(\sigma_1)} \xi(\sigma_1), \xi(\sigma_1) \rangle - \langle \overline{U}_a^{(\sigma_2)} \xi(\sigma_2), \xi(\sigma_2) \rangle \ U_a^{(\sigma_1)} \xi(\sigma_1), \xi(\sigma_1) \rangle \ da \right| \leq \varepsilon,
\]

while the first member has the value $\frac{1}{d_{\sigma_1}}$. Since $d_{\sigma_1} \leq n$ (fixed), we arrive at a contradiction by our choice of $\varepsilon$. $\blacksquare$

3. CONNECTION OF $\tau_c$ WITH OTHER TOPOLOGIES ON $L_\infty(G)$.

From Proposition 1 we immediately conclude that for non-compact $G$ the $w^*$-topology is always strictly weaker than the $\tau_c$-topology. But taking Proposition 2 into account, we infer that also for infinite compact $G$ these two topologies are different. Indeed, it suffices to remark that there always exists a function $f$
in $L_1(G)$ which is not a trigonometric polynomial (e.g., choose in $\sum_1^n$ a countable infinite set $\{\sigma_n\}_{n=1}^\infty$ of different elements; let $\chi_{\sigma_n}$ be the corresponding character, and put $f(x) = \sum_{n=1}^\infty \frac{\chi_{\sigma_n}(x)}{n^2}$ for $x \in G$; then $f \in L_1(G)$, and $f(\sigma_n) = \frac{1}{n^2} I_{H_n}$, where $I_{H_n}$ is the identity operator on $H_n$.

Although $\tau_C$ and $\omega^*$ are different for infinite compact $G$, they induce the same topology on every norm-bounded subset of $L_\infty(G)$, as the following proposition shows.

**PROPOSITION 3.** If $G$ is compact, and $B$ is a norm-bounded subset of $L_\infty(G)$, then $\tau_C$ and $\omega^*$ coincide on $B$.

**PROOF.** It is sufficient to prove that for any $\tau_C$-neighborhood $V$ of $0$ there exists a $\omega^*$-neighborhood $W$ of $0$ such that $W \cap B \subseteq V$. Suppose that $\|h\|_\infty < M$, $h \in B$, and let $V = \{h \in L_\infty(G) : \|f_i \cdot h\|_\infty < \epsilon, \text{ for } i = 1, \ldots, n\}$ with given $f_i \in L_1(G)$ and $\epsilon > 0$. From compactness of $G$ and continuity of $a^* f$ from $G$ to $(L_1(G), \| \cdot \|_1)$ it follows that each $f_i$ is almost periodic in $(L_1(G), \| \cdot \|_1)$; this means that there exists elements $a_1, \ldots, a_m$ in $G$ such that, for each $a$ in $G$ and each $i \in \{1, \ldots, n\}$ a point $j(i)$ may be found ($1 \leq j \leq m$) such that $\| f_i - a_j \cdot f_i \| < \epsilon$. With this choice of $a_j$ and for $g \in L_\infty(G)$ we have

$$\| (f_i \cdot g)(a) - (f_i \cdot g)(a_j) \| < \epsilon, \quad \| a_j - a_i \| \cdot \| g \|_\infty,$$

or for $g \in B$, $\| (f_i \cdot g)(a_j) \| < \epsilon$. Then $W = \omega^*$-neighborhood of $0$, and for $h$ in $W \cap B$ we obtain $\| f_i \cdot h \|_\infty < \epsilon$.

**COROLLARY 1.** For compact $G$, any $\omega^*$-convergent sequence is $\tau_C$-convergent. Indeed, the set consisting of the elements in the sequence together with its limit is $\omega^*$-compact, and hence also norm bounded.

**COROLLARY 2.** For compact $G$, $L_1(G)$ is the dual of $(L_\infty(G), \tau_C)$.

**PROOF.** For a compact group $G$ there is a connection between the $\tau_C$-topology and the so-called bounded weak*-topology $b\omega^*$ (see Holmes [9], p. 150; this topology is called the bounded $X$-topology in Dunford and Schwartz [10], p. 427); indeed, we have $\tau_C \leq b\omega^*$. The result then follows from the fact that $L_1(G)$ is the dual of $(L_\infty(G), b\omega^*)$.

The following proposition characterizes those groups for which $\tau_C$ and $\| \cdot \|_\infty$
PROPOSITION 4. \( \tau_c \) coincides with \( || ||_\infty \) iff \( G \) is discrete.

PROOF. For discrete \( G \) we have that the \( \tau_c \)-topology and the \( || ||_\infty \)-topology are equal. For if \( e \) is the identity of \( G \), then \( \delta_e \) is a convolution identity for \( L_1(G) \), and the convolution operator \( T_{\delta_e} \) induced by \( \delta_e \) is the identity map on \( L_\infty(G) \).

Let then \( G \) be non-discrete. Given \( \varepsilon > 0 \) and \( f_1, \ldots, f_n \) in \( L_1(G) \), choose a compact subset \( K \) in \( G \) such that \( \int |f_i(x)| dx < \frac{\varepsilon}{2^n} \), for each \( i = 1, \ldots, n \). Let \( \eta > 0 \) be such that \( G/K \)
\[
\int |f_i(x)| dx < \frac{\varepsilon}{2^n} \quad \text{for all } i = 1, \ldots, n \text{ and all measurable } A \subseteq G \text{ with } \mu(A) < \eta, \text{ where } \mu \text{ denotes left Haar measure.}
\]
Further, let \( U \) be a compact symmetric neighborhood of the identity \( e \) of \( G \) with \( 0 < \mu(U) < \eta \). Let \( g \) be the function defined by \( g(x) = 1 \) for \( x \in U \), and \( g(x) = 0 \) on \( G \setminus U \). For \( 1 \leq i \leq n \) and \( x \in G \) we obtain
\[
|g(x)| \leq \int_{G \setminus K} |f_i(y)| dy + \int_K |f_i(y)g(y^{-1}x)| dy.
\]
Both terms on the right-hand side are dominated by \( \frac{\varepsilon}{2^n} \), since \( g(y^{-1}x) \) is zero except when \( y \in xU^{-1} \), and since \( \mu(xU^{-1}) < \eta \). Hence \( ||f_1 \ast g||_\infty < \varepsilon \), although \( ||g||_\infty = 1 \). This shows that no \( \tau_c \)-neighborhood of \( 0 \) lies wholly in any \( || ||_\infty \)-ball of radius less than \( 1 \). Hence \( \tau_c \) is coarser than \( || ||_\infty \).

4. FURTHER PROPERTIES OF THE \( \tau_c \)-TOPOLOGY.

The proofs of Propositions 5 and 8 that follow were kindly suggested to us by Robert B. Burckel. Both results also appear in Crombez and Govaerts[2].

PROPOSITION 5. Any norm-closed ball in \( L_\infty(G) \) is \( \tau_c \)-complete.

PROOF. Let \( \{g_\alpha\} \) be a \( \tau_c \)-Cauchy net in a ball in \( L_\infty(G) \). Let \( g \) be a \( \omega^\ast \)-cluster point of this net, such that a subnet \( \{g_\beta\} \) \( \omega^\ast \)-converges to \( g \). Then \( \{(f \ast g_\beta)(x)\} \) converges to \( (f \ast g)(x) \) for all \( x \) in \( G \) and all \( f \) in \( L_1(G) \). Given \( \varepsilon > 0 \) and \( f \in L_1(G) \), there exists \( \alpha_\varepsilon \) such that \( ||f \ast g_\beta - f \ast g_\alpha||_\infty \leq \varepsilon \) for all \( \alpha, \alpha' \geq \alpha_\varepsilon \). Since all these functions are continuous and \( || ||_\infty \) here is genuine supremum, we derive
\[
|(f \ast g_\alpha)(x) - (f \ast g_\alpha')(x)| \leq \varepsilon \quad \text{for all } \alpha, \alpha' \geq \alpha_\varepsilon, \text{ for all } x \in G.
\]
In this last inequality
we take \( \alpha' = \beta \) and let \( \beta \) recede to infinity; then this leads to
\[
| (f g_{\alpha}(x) - (f g)(x)) | < \varepsilon \text{ for all } \alpha \gg \alpha_0 \text{ and all } x \in G, \text{ i.e.,}
\]
\[
| f g_{\alpha} - f g |_\infty < \varepsilon \text{ for all } \alpha \gg \alpha_0.
\]

In particular, we derive from Proposition 5 that a set in \( L_{\infty}(G) \) is \( \tau_c \)-relatively compact iff it is \( \tau_c \)-totally bounded. We also have that the closed absolutely convex hull of a \( \tau_c \)-compact set is again \( \tau_c \)-compact. Denoting by \( \text{cl} \tau_c \) the closure in the \( \tau_c \)-topology, we have

**Proposition 6.** If \( g \in L_{\infty}(G) \), then \( g \in \text{cl} \tau_c (L_1 g) \).

**Proof.** Given \( \varepsilon > 0 \) and \( n \) functions \( k_i \) in \( L_1(G) \) determining a \( \tau_c \)-neighborhood \( V \) of \( g \) in \( L_{\infty}(G) \), and denoting by \( \{ e_\lambda \} \) an approximate identity in \( L_1(G) \), we see that
\[
|| k_i(e_\lambda g) - k_1 g ||_\infty \text{ may be made arbitrarily small. Hence } V \text{ contains elements of the form } e_\lambda g.
\]

**Corollary 3.** Let \( S \) be a \( \tau_c \)-closed \( L_1 \)-Submodule of \( L_{\infty}(G) \). Then \( S = \text{cl} \tau_c (L_1 S) \).

**Corollary 4.** Let \( S \) be a \( \tau_c \)-closed \( L_1 \)-Submodule of \( L_{\infty}(G) \). Then \( S \) is left translation invariant.

**Proof.** Given \( g \in S \) and \( a \in G \) we show that any \( \tau_c \)-neighborhood of \( a g \) contains a function in \( L_1 S \), from which the result will follow. Denote by \( \Delta \) the modular function of \( G \). Let \( V \) be the \( \tau_c \)-neighborhood of \( a g \) determined by \( f_1, \ldots, f_n \) in \( L_1(G) \) and \( \varepsilon > 0 \). There always exist \( k \in L_1(G) \) and \( h \in S \) such that
\[
|| (f_1 a g - f_1 a)(kh) ||_\infty < \frac{\varepsilon}{\Delta(a)}. \text{ Then } || f_1 a g - f_1 a(kh) ||_\infty < \varepsilon. \text{ Hence } V \text{ contains the function } a(kh) \in L_1 S.
\]

Since \( \mathbf{w} < \tau_c \), Proposition 6 and its corollaries are stronger than the corresponding results in Crombez and Govaerts [5].

Given \( g \in L_{\infty}(G) \), the map \( s \mapsto s g \) from \( G \) to \( (L_{\infty}(G), || \cdot ||_\infty) \) is continuous iff \( g \) is locally a.e. equal to a function in \( C_{\text{ru}}(G) \). (Here, as in [11], \( C_{\text{ru}}(G) \) is the set of all right uniformly continuous, bounded, complex-valued functions on \( G \)). However, using the \( \tau_c \)-topology on \( L(G) \) we obtain continuity for any \( g \in L_{\infty}(G) \).
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PROPOSITION 7. Let \( g \) be a function in \( L^\infty(G) \). Then the maps \( s \mapsto g_s \) and \( s \mapsto g \) from \( G \) to \( (L^\infty(G), \tau_c) \) are continuous.

**Proof.** That the map \( s \mapsto g_s \) is continuous is trivial, since for any \( f \) in \( L^1(G) \), \( f g \in C_c(G) \) and \( f g_s = (f g)_s \). To prove that \( s \mapsto g \) is continuous, consider the composition of the maps \( G \rightarrow L^1(G) \times C_c(G) \) given by \( s \mapsto (f_s, \Delta(s)) \rightarrow \Delta(s)f_s g = f_s g \).

Each map is continuous, and so the result follows. □

5. SOME MORE RESULTS ON \( \tau_c \)-ALMOST PERIODIC FUNCTIONS.

In this final section we always suppose \( G \) to be Abelian. The notion of \( \tau_c \)-almost periodic (\( \tau_c \)-AP) function in \( L^\infty(G) \) was introduced in [2] in order to characterize those multipliers which are strongly almost periodic.

**Proposition 8.** A function \( g \) in \( L^\infty(G) \) is \( \tau_c \)-AP iff \( f g \) is \( \| \|_\infty \)-almost periodic for each \( f \) in \( L^1(G) \).

**Proof.** We first notice that \( f g_a = (f g)_a \) for any \( a \) in \( G \); so if we set \( 0_g = \{ g_a : a \in G \} \), then \( f g_0 = 0_f g \).

If \( 0_g \) is relatively \( \tau_c \)-compact, then its continuous image \( f g_0 = 0_f g \) in \( (C_c(G), \| \|_\infty) \) is relatively compact, so \( f g \) is norm almost periodic. Conversely, by definition of \( \tau_c \) the map

\[
\begin{align*}
g \mapsto (f g)_f & \in L^\infty(G) \\
& \quad \text{for } f \in L^1(G), \quad g \in L^1(G)
\end{align*}
\]

is a homeomorphism from \( \tau_c \) into the product of the norm topologies on the right. Evidently the image of \( 0_g \) lies in the subspace \( \| \|_\infty \)-almost periodic, if \( f g \) is norm almost periodic, then this last product is relatively compact, and so \( 0_g \) is relatively \( \tau_c \)-compact. □

Denoting by \( AP \) the \( \| \|_\infty \)-almost periodic functions in \( L^\infty(G) \), we obtained in [2] that \( \tau_c \)-AP = \( AP \) for \( G \) discrete, and \( \tau_c \)-AP = \( L^\infty(G) \) for \( G \) compact (both results are of course clear now by Proposition 4 and Proposition 3, respectively). We always have that \( AP \subseteq \tau_c \)-AP. From Proposition 8 we derive: \( L^1(G) \tau \)-AP = \( AP \). Since \( L^1(G) \tau \)-AP = \( AP \) (see Crombez and Govaerts [4]), we also get \( L^1(G) \tau \)-AP = \( AP \). Hence we obtain from Proposition 8 that \( \tau_c \)-AP is the largest linear subspace \( S \) of \( L^\infty(G) \).
such that \( L^1(G) \star S = AP \). The set \( \tau_c^{-1}AP \) is an \( L_1 \)-submodule of \( L_\infty(G) \) which is obviously \( \tau_c \)-closed. From Corollary 3 we may conclude that \( \tau_c^{-1}AP = cl_{\tau_c} AP \). In particular, for compact \( G \) we have that \( L_\infty(G) = cl_{\tau_c} C(G) \), where \( C(G) \) denotes the set of continuous functions on \( G \).

**PROPOSITION 9.** \( G \) is compact iff \( \tau_c^{-1}AP = L_\infty(G) \).

**PROOF.** Suppose that \( \tau_c^{-1}AP = L_\infty(G) \). Then \( AP = L_1(G) \star \tau_c^{-1}AP = L_1(G) \star L_\infty(G) = C_{ru}(G) \), the last equality coming from Hewitt and Ross [6], 32.45(b). Pick \( 0 \neq f \in C_{ru}(G) \) with compact support \( K \). If \( G \) is not compact there exist infinitely many disjoint translates \( a \cdot K \) of \( K \). Clearly the subset \( \{ -l^{-1}f \}^\infty_{j=1} \) of the left orbit of \( f \) is not totally bounded.\( \blacksquare \)

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