PARTIAL HENSELIZATIONS

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ABSTRACT. We define and note some properties of k H-pairs (k Henselian pairs), k N-pairs, and k N'-pairs. It is shown that the 2-Henselization and the 3-Henselization of a pair exist. Characterizations of quasi-local 2H-pairs are given, and an equivalence to the chain conjecture is proved.

KEY WORDS AND PHRASES. k Henselian pair, k N-pair, k N'-pair, chain conjecture.

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1. INTRODUCTION.

We define a pair (A,m) to be a k H-pair (a k Henselian pair) in case the ideal m is contained in the Jacobson radical of the commutative ring A and if for every monic polynomial f(X) of degree k in A[X] such that \( \overline{f}(X) \in A/m [X] \) factors into \( \overline{f}(X) = \overline{g}_o(X)\overline{h}_o(X) \) where \( \overline{g}_o(X) \) and \( \overline{h}_o(X) \) are monic and coprime, there exist monic polynomials g(x), h(X) \( \in A[X] \) such that f(X) = g(X)h(X), \( \overline{g}(X) = \overline{g}_o(X) \), and \( \overline{h}(X) = \overline{h}_o(X) \). It is shown that the 2-Henselization and the 3-Henselization of a pair (A,m) exist. Several properties of k H-pairs are noted. And an equivalence to the Chain Conjecture is also given.

2. k H-PAIRS, k N-PAIRS, AND k N'-PAIRS.

In this section we define and give some facts about k H-pairs, k N-pairs, and
The main result, Theorem (2.10) states that (i) a \( k \) \( H\)-pair is a \( k \)
\( N\)-pair, (ii) a \( k \) \( N\)-pair is a \( k \) \( N'\)-pair, and (iii) an \( k \) \( N'\)-pair is a \( j \) \( H\)-pair provided \( k \geq \max \{c_{j,n} \mid n = 0,1,...,j\} \).

We begin by stating several definitions. In these definitions and throughout
the paper a ring shall mean a commutative ring with an identity element, and \( J(A) \)
denotes the Jacobson radical of the ring \( A \).

**Definition 2.1.** \((A,m)\) is a pair in case \( A \) is a ring and \( m \) is an ideal in \( A \).

**Definition 2.2.** \((A,m)\) is a \( k \) \( H\)-pair in case

(i) \( m \subset J(A) \); and

(ii) for every monic polynomial \( f(X) \) of degree \( k \) in \( A[X] \) such that
\( f(X) \in A/m[X] \) factors into \( \bar{f}(X) = \bar{g}_0(X) \bar{h}_0(X) \) where \( \bar{g}_0(X) \) and \( \bar{h}_0(X) \) are monic and
coprime, there exist monic polynomials \( g(X), h(X) \in A[X] \) such that \( f(X) = g(X)h(X) \),
\( \bar{g}(X) = \bar{g}_0(X) \) and \( \bar{h}(X) = \bar{h}_0(X) \).

**Definition 2.3.** Let \((A,m)\) be a pair. A monic polynomial \( X^k + a_{k-1}X^{k-1} + ... + a_1X + a_0 \) of degree \( k \) is called a \( k \) \( N\)-polynomial over \((A,m)\) in case \( a_d \in m \) and
\( a_1 \) is a unit mod \( m \).

**Definition 2.4.** \((A,m)\) is a \( k \) \( N\)-pair in case

(i) \( m \subset J(A) \); and

(ii) every \( k \) \( N\)-polynomial over \((A,m)\) has a root in \( m \).

The next results give some facts about \( k \) \( N\)-polynomials and \( k \) \( N\)-pairs.

**Lemma 2.5.** Let \( f(X) \) be a \( k \) \( N\)-polynomial over the pair \((A,m)\). If \( m \subset J(A) \),
then \( f(X) \) has at most one root in \( m \).

**Proof.** The proof follows from [5, Lemma 1.5], since a \( k \) \( N\)-polynomial is an
\( N\)-polynomial.

**Remark.** Every \( k \) \( N\)-polynomial over a \( k \) \( N\)-pair \((A,m)\) has one and only one root
in \( m \).

**Proposition 2.6.** If \((A,m)\) is a \( k \) \( N\)-pair, then \((A,m)\) is an \( j \) \( N\)-pair for 2
\( 2 \leq j \leq k \).

**Proof.** Given a \( k \) \( N\)-pair \((A,m)\), it suffices to show that \((A,m)\) is a \((k-l)\)
\( N\)-pair. Let \( f(X) \) be a \((k-l)\) \( N\)-polynomial over \((A,m)\). Let \( u \) be a unit in \( A \) and
\( g(X) = (X + u)f(X) \). Then \( g(X) \) is a \( k \) \( N \)-polynomial and thus has a root \( r \) in \( m \) and 
\( 0 = g(r) = (r + u)f(r) \). Since \( r + u \) is a unit, we have \( f(r) = 0 \). Therefore, 
\( (A, m) \) is a \( (k - 1) \) \( N \)-pair.

**Definition 2.7.** Let \( (A, m) \) be a pair. A monic polynomial
\[ x^k + d_1 x^{k-1} + d_2 x^{k-2} + \ldots + d_k \]
of degree \( k \) is called a \( k \) \( N' \)-polynomial over \( (A, m) \) in case \( d_1 \) is a unit mod \( m \) and \( d_2, \ldots, d_k \) belong to \( m \).

**Definition 2.8.** \( (A, m) \) is a \( k \) \( N' \)-pair in case

(i) \( m \subseteq J(A) \); and

(ii) every \( k \) \( N' \)-polynomial over \( (A, m) \) has a root in \( A \), which is a unit.

We note that if \( (A, m) \) is a \( k \) \( N' \)-pair, \( f(X) = x^k + d_1 x^{k-1} + \ldots + d_k \) is a \( k \) \( N' \)-polynomial over \( (A, m) \) and \( r \in A \) is a root of \( f(X) \) given by the definition of a \( k \) \( N' \)-pair, then \( r = -d_1 \), and \( f'(r) \) is a unit.

**Proposition 2.9.** Let \( (A, m) \) be a \( k \) \( N' \)-pair, then \( (A, m) \) is an \( j \) \( N' \)-pair for \( 2 \leq j \leq k \).

**Proof.** Given a \( k \) \( N' \)-pair \( (A, m) \), it suffices to show that \( (A, m) \) is a \( (k-1) \) \( N' \)-pair. Let \( f(X) \) be a \( (k-1) \) \( N' \)-polynomial over \( (A, m) \). Then \( Xf(X) \) is a \( k \) \( N' \)-polynomial and has a root \( u \), which is a unit. and \( uf(u) = 0 \) implies that \( f(u) = 0 \), therefore \( (A, m) \) is a \( (k-1) \) \( N' \)-pair.

**Theorem 2.10.** (i) A \( kH \)-pair is a \( kN \)-pair

(ii) A \( kN \)-pair is a \( kN' \)-pair

(iii) A \( kN' \)-pair is a \( jH \)-pair, provided
\[ k \geq \max \{ C_{j, n} \mid n = 0, 1, \ldots, j \} \]

**Proof.** Part (i) follows from the definitions.

The proof of (ii) follows from the proof of [10, Lemma 7]

The proof of (iii) follows from Crépeaux's proof of [3, Prop. 1]

3. \( k \) \( N \)-Closure.

In this section we construct the \( k \) \( N \)-closure for a given pair \( (A, m) \). That is, we find the "smallest" \( k \) \( N \)-pair which "contains" \( (A, m) \). The development of this section parallels Greco's development in [5].

In order to construct the \( k \) \( N \)-closure we need the following definitions.
DEFINITION 3.1. A morphism (of pairs) $\phi: (A,m) \to (B,n)$ is a ring homomorphism $\phi: A \to B$, such that $\phi^{-1}(n) = m$.

DEFINITION 3.2. A morphism (of pairs) $\phi: (A,m) \to (B,n)$ is strict in case $n = \phi(m)B$ and $\phi$ induces an isomorphism $A/m \to B/n$.

DEFINITION 3.3. Let $(A,m)$ be a pair. A $k$-pair $(B,n)$ together with a morphism $\phi: (A,m) \to (B,n)$ is a $k$-closure of $(A,m)$ if for any $k$-pair $(B',n')$ and any morphism $\psi: (A,m) \to (B',n')$ there exists a unique morphism $\psi': (B,n) \to (B',n')$ such that $\psi' \circ \phi = \psi$.

DEFINITION 3.4. Let $(A,m)$ be a pair and $f(X)$ a $k$-polynomial over $(A,m)$. Let $A[x] = A[X]/(f(X))$, $S = 1 + (m,x)A[x]$ and $B = S^{-1}A[x]$. Then $(B,mB)$ is called a simple $k$-extension of $(A,m)$.

DEFINITION 3.5. A $k$-extension of $(A,m)$ is a pair obtained from $(A,m)$ by a finite number of simple $k$-extensions.

The next two results give some useful properties of simple $k$-extensions and $k$-extensions.

LEMMA 3.6. Let $(B,n)$ be a simple $k$-extension of $(A,m)$. Let $\phi: A \to B$ be the canonical morphism. Then:

(i) $x \in n$.

(ii) $\phi^{-1}(n) = m$ and $\phi: (A,m) \to (B,n)$ is a morphism of pairs.

(iii) $\phi: (A,m) \to (B,n)$ is strict.

PROOF. The proof follows from [5, Lemmas 2.3, 2.4, and 2.5] since a simple $k$-extension is a simple $N$-extension.

COROLLARY 3.7. If $(B,n)$ is a $k$-extension of $(A,m)$, then the canonical morphism $\phi: (A,m) \to (B,n)$ is strict.

We note that a $k$-extension of a quasi-local ring $(A,m)$ is a quasi-local ring.

The following lemma is used to show that the partial order defined in Definition (3.9) is well defined.

LEMMA 3.8. Let $(A',m')$ be a $k$-extension of $(A,m)$ and let $(B,n)$ be a pair with $n \subseteq J(B)$. Let $\phi: (A,m) \to (A',m')$ be the canonical morphism. Then for any
morphism \( \psi: (A, m) \to (B, n) \) there is at most one morphism \( \psi': (A', m') \to (B, n) \) such that \( \psi' \circ \emptyset = \psi \).

**PROOF.** The proof follows from [5, Lemma 3.1] since a \( k \)-\( N \)-extension is an \( N \)-extension.

In particular, the above lemma holds when \((B, n)\) is a \( k \)-\( N \)-extension of \((A, m)\).

**DEFINITION 3.9.** Define a partial order on the set of \( k \)-\( N \)-extensions of \((A, m)\) as follows: If \((A', m')\) and \((A'', m'')\) are two \( k \)-\( N \)-extensions of \((A, m)\), then \((A', m') \leq (A'', m'')\) if and only if there is a morphism \( \psi: (A', m') \to (A'', m'') \) such that \( \psi \circ \emptyset = \emptyset '' \), where \( \emptyset: (A, m) \to (A', m') \) and \( \emptyset '': (A, m) \to (A'', m'') \) are the canonical morphisms.

**PROPOSITION 3.10.** Let \((A, m)\) be a pair. Then the \( k \)-\( N \)-extensions of \((A, m)\) form a directed set with the order relation and the morphisms defined above.

**PROOF.** The proof is analogous to [5, Prop. 3.3].

**LEMMA 3.11** Let \((A', m')\) be a \( k \)-\( N \)-extension of \((A, m)\) and let \( \emptyset: (A, m) \to (A', m') \) be the canonical morphism. Let \((B, n)\) be a \( k \)-\( N \)-pair and let \( \psi: (A, m) \to (B, n) \) be a morphism. Then there is a unique morphism \( \psi': (A', m') \to (B, n) \) such that \( \psi = \psi' \circ \emptyset \).

**PROOF.** The proof is analogous to [5, Prop. 3.4].

**THEOREM 3.12.** Let \((A, m)\) be a pair and let \((A^{kN}, m^{kN})\) be the direct limit of the set of all \( k \)-\( N \)-extensions. Then \((A^{kN}, m^{kN})\) with the canonical morphism \((A, m) \to (A^{kN}, m^{kN})\) is a \( k \)-\( N \)-closure of \((A, m)\).

**PROOF.** The proof is analogous to [5, Thm. 3.5].

We note that if \((A, m)\) is a quasi-local ring; then a \( k \)-\( N \)-closure \((A^{kN}, m^{kN})\) of \((A, m)\) is quasi-local, since the direct limit of quasi-local rings is quasi-local.

4. **\( k \)-\( H \)-CLOSURES AND AN EQUIVALENCE TO THE CHAIN CONJECTURE.**

In this section, we note the existence of a \( 2H \)-closure and of a \( 3H \)-closure, we give some characterization of a quasi-local \( 2H \)-pair, and we observe that the \( H \)-closure (or Henselization) of a pair \((A, m)\) can be written as the direct limit or union of \( k \)-\( H \)-pairs, \( k = 2, 3, 4, \ldots \). We also give an equivalence to the Chain Conjecture.

**DEFINITION 4.1.** Let \((A, m)\) be a pair. A \( k \)-\( H \)-pair \((B, n)\), together with a
morphism $\phi:(A,m)\to(B,n)$ is a $k$-H-closure of $(A,m)$ if for any $k$-H-pair $(B',n')$ and any morphism $\psi:(A,m)\to(B'n')$, there exists a unique morphism $\psi':(B,n)\to(B',n')$ such that $\psi'\circ\phi=\psi$.

THEOREM 4.2. Let $(A,m)$ be a pair. Then:

(i) a 2-H-closure of $(A,m)$ is $(A^{2N}, m^{2N})$.

(ii) a 3-H-closure of $(A,m)$ is $(A^{3N}, m^{3N})$.

PROOF. It suffices to show that a $k$-N-closure $(k=2,3)$ is a $k$-H-pair. And by Theorem 2.10, we have that a $2N$-pair is a 2H-pair, and that a $3N$-pair is a 3H-pair.

DEFINITION 4.3. If $\phi:A+B$ is a ring homomorphism, then $B$ is said to be $k$-integral over $A$ in case each $b \in B$ satisfies a monic polynomial of degree $k$ over $\phi(A)$.

REMARK. If $B$ is $k$-integral over $A$, then $B$ is also $j$-integral over $A$ for all $j \geq k$.

In the next three items we give examples of rings and elements which are $k$-integral over a given ring $A$.

LEMMA 4.4. If $A$ is an integrally closed domain and $f(X) \in A[X]$ is a monic polynomial of degree $k$, then $A[X]/(f(X))$ is $k$-integral over $A$.

PROOF. Let $A[x] = A[X]/(f(X))$ and let $L$ be the quotient field of $A$. Then $[L(x):L] \leq k$ and thus each $\alpha \in A[x]$ satisfies a monic polynomial $g(X) \in L[X]$ of degree $\leq k$. Since $\alpha$ is integral over $A$ and $A$ is integrally closed, it follows that $g(X) \in A[X]$. Therefore $A[x]$ is $k$-integral over $A$.

LEMMA 4.5. Let $A$ be a ring and let $f(X) = X^2 + \alpha X + \beta \in A[X]$. Then $A[X]/(f(X))$ is 2-integral over $A$.

PROOF. Let $A[x] = A[X]/(f(X))$ and then all of the elements of $A[X]$ are of the form $ax + b$ where $a,b \in A$. To show that $A[x]$ is 2-integral over $A$, we need to find $F,G \in A$ such that

$$(ax + b)^2 + F(ax + b) + G = 0.$$ 

By expanding the left side, we see that $F = a\alpha - 2b$ and $G = a^2\alpha - b^2 - Fb = a^2\beta + b^2 - ab\alpha$ are the needed values. Therefore $A[X]$ is 2-integral over $A$.

EXAMPLE 4.6. Each element of $\text{End}_A(A^k)$ is $k$-integral over $A$ by [1, Proposition 2.4].
In fact, if $M$ is any $A$-module generated by $k$ elements, each element of $\text{End}_A(M)$ is $k$-integral over $A$.

**Definition 4.7.** $(A,m)$ is a $(\leq k)H$-pair in case $(A,m)$ is a $jH$-pair for $2 \leq j \leq k$.

It follows by Theorem 2.10 that if $(A,m)$ is a $jN$-pair (or $jH$-pair), then $(A,m)$ is a $(\leq k)H$-pair provided $j \geq \max\ {C_k,n \mid n = 0, 1, \ldots k}$. In particular we have that for $k = 2,3$, or $4$, a $kH$-pair is also a $(\leq k)H$-pair.

**Lemma 4.8.** Let $(A,m)$ be a quasi-local domain which is a $(\leq k)H$-pair. Then every $k$-integral extension domain of $A$ is quasi-local.

**Proof.** The proof is analogous to [6, (30.5)]

**Definition 4.9.** A ring $A$ is decomposed if $A$ is the product of finitely many quasi local rings.

**Theorem 4.10.** Let $(A,m)$ be a quasi local ring. Then the following statements are equivalent.

(i) Every finite 2-integral $A$-algebra $B$ is decomposed.

(ii) Every finite free 2-integral $A$-algebra $B$ is decomposed.

(iii) Every $A$-algebra of the form $A[X]/(f(X))$, where $f(X) \in A[X]$ is monic and of degree 2, is decomposed.

(iv) $(A,m)$ is a $2H$-pair.

**Proof.** (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) is clear by (4.5). The proofs that (iii) $\Rightarrow$ (i) and that (iii) $\Leftrightarrow$ (iv) follow classical lines; for example, see [9, Prop. 5, p.2].

**Theorem 4.11.** A quasi local domain $(A,m)$ is a $2H$-pair if and only if every 2-integral extension domain $A'$ of $A$ is quasi-local.

**Proof.** ($\Rightarrow$) is true by (4.8).

($\Leftarrow$). We will show that $(A,m)$ is a $2H$-pair by showing that every finite free 2-integral $A$-algebra is decomposed. Let $B$ be a finite free 2-integral $A$-algebra. Since $B$ is decomposed if and only if $B/\text{nil \, rad \, B}$ is decomposed, we may assume that $B$ is reduced. Since $B$ is flat over $A$, regular elements of $A$ are also regular in $B$. Thus the minimal primes of $B$ contract to $\{0\}$ in $A$. Let $\{P_i \mid i \in I\}$ be the minimal primes of $B$. Then for each $i \in I$, $B/P_i$ is a 2-integral extension domain of $A$ and is quasi local by the hypothesis. Thus each minimal prime $P_i$ is contained in a unique maximal
ideal. By [2, Proposition 3, p. 329], the set of minimal primes of $B$ is finite. Let $I_j = \cap_{i \in M_j} P_i$, where $M_j$, $j=1,\ldots, n$, are the maximal ideals of $B$. Then the $I_j$ are coprime, and $\cap_{j=1}^n I_j = 0$ since $B$ is reduced. So by the Chinese Remainder Theorem $B \cong \prod_{j=1}^n B/I_j$ and each $B/I_j$ is quasi local. Thus $B$ is decomposed and therefore $(A,m)$ is a $2H$-pair.

**COROLLARY 4.12.** Let $(A,m)$ be a quasi local domain which is $2H$-pair. Let $A'$ be an integral extension domain of $A$. If $b \in A'$ is 2-integral over $A$, then $b \in J(A')$ or $b$ is a unit.

**PROOF.** $A[b]$ is a 2-integral extension domain of $A$ and is thus quasi local. The result follows since all the maximal ideals of $A'$ contract to the unique maximal ideal of $A[b]$.

We will now show that the $N$-closure of a pair $(A,m)$ is the direct limit of the $k$ $N$-closures of $(A,m)$. It will follow from this result that the $H$-closure of $(A,m)$ can be written as the direct limit of $k$ $H$-pairs.

**DEFINITION 4.13.** Let $(A,m)$ be a pair. Then $(A,m)$ is an $N$-pair (respectively, a $H$-pair) in case $(A,m)$ is a $k$ $N$-pair (respectively, a $k$ $H$-pair) for $k = 2,3,\ldots$.

**DEFINITION 4.14.** Let $(A,m)$ be a pair. An $N$-pair (respectively, an $H$-pair) $(B',n')$, together with a morphism $\phi:(A,m)\rightarrow(B',n')$ is an $N$-closure (respectively, an $H$-closure) of $(A,m)$ if for any $N$-pair (respectively, any $H$-pair) $(B',n')$, and any morphism $\psi':(A,m)\rightarrow(B',n')$, there exists a unique morphism $\psi:(B,n)\rightarrow(B',n')$ such that $\psi' \circ \phi = \psi$.

**THEOREM 4.15.** Let $(A,m)$ be a pair. Then the $H$-closure of $(A,m)$ is isomorphic to the $N$-closure.

**PROOF.** See [5, Lemma 1.4 and Theorem 5.10].

**PROPOSITION 4.16.** Let $(A^N,m^N)$ be an $N$-closure of $(A,m)$. Then $(A^N,m^N) \sim \operatorname{dir lim} (A^{kN},m^{kN})$, where the directed system $\{(A^{kN},m^{kN}),\mu_{kj}\}$ of $k$ $N$-closures of $(A,m)$, $k=2,3,\ldots$, is ordered by $(A^{kN},m^{kN}) \leq (A^{jN},m^{jN})$ iff $k \leq j$ and if $k \leq j$, then $\mu_{kj}:(A^{kN},m^{kN})\rightarrow(A^{jN},m^{jN})$ is the unique morphism which makes the following diagram commute:
where \( \emptyset_j \) and \( \emptyset_k \) are the canonical morphisms.

**Proof.** The proof follows immediately from Definitions (3.3) and (4.14) and the definition of a direct limit.

**Corollary 4.17.** Let \( (A^H, m^H) \) be the \( H \)-closure of \( (A, m) \). Then \( (A^H, m^H) \) is
\[
\text{dir lim } (A_j, m_j)_i \quad \text{where } (A_j, m_j)_i \text{ is an } i - \text{pair for } i = 2, 3, \ldots.
\]

**Proof.** For a given \( i \), let \( (A_j, m_j)_i = (A_{kN}, m_{kN}) \) where \( k = \max \{ C_j, n=0, 1, \ldots, j \} \).

Then the corollary follows by results (2.10), (4.15) and (4.16).

We now give an equivalence to the Chain Conjecture. The terminology used is the same as in [8] or [10].

**Theorem 4.18.** The following statements are equivalent:

(i) The Chain Conjecture holds.

(ii) Every \( 2 \)-Henselian local domain \( A \), such that the integral closure of \( A \) is quasi-local, is catenary.

**Proof.** (i) \( \Rightarrow \) (ii). This follows by [8, Thm. 2.4].

(ii) \( \Rightarrow \) (i). By [8, Thm. 2.4] it suffices to show that every Henselian local domain is catenary. Let \( A \) be a Henselian local domain. Then \( A \) is also \( 2 \)-Henselian and the integral closure of \( A \) is quasi-local by [6, (43.12)]. Thus by the hypothesis \( A \) is catenary.

5. **Examples.**

In this section we show that there exist \( k \)-N-pairs which are not N-pairs and there exist \( k \)-H-pairs which are not H-pairs. More precisely, for each prime number \( p \) we give an example of a pair which is not a \( p \)-N-pair but is a \( k \)-N-pair for \( 2 \leq k < p \). This example also shows that for any integer \( k \geq 2 \), there exists a \( k \)-H-pair which is not a \( p \)-H-pair for some sufficiently large prime number \( p \).

Let \( p > 2 \) be a prime number. Let \( (R, q) \) be a normal quasi-local domain such that there exists an \( f(X) = X^p + \ldots + a_1X + a_0 \in R[X] \), where \( a_1 \notin q \), \( a_0 \in q \) and \( f(X) \)
is irreducible over \(R[X]\).

In particular, let \(R = \mathbb{Z}(2)\) and let \(f(X) = X^p + 3X + 6\). Then by Eisenstein's Criterion, \(f(X)\) is irreducible in \(Q[X]\), and thus irreducible in \(\mathbb{Z}(2)[X]\) since \(f(X)\) has content 1.

Let \(K\) be the quotient field of \(R\) and let \(\overline{K}\) be an algebraic closure of \(K\). Let \(R'\) be the integral closure of \(R\) in \(\overline{K}\) and \(P'\) any maximal ideal in \(R'\). Now \(f(X)\) as an element of \(R'[X]\) factors completely, and since \(P' \cap R = q\), \(f(X)\) has a unique root \(\alpha \in P'\). Let \(L\) be the least normal extension of \(K\) containing \(\alpha\). Then \(p\mid [L:K]\) and by [7, Thm. 6] there is a maximal field \(M\) without \(\alpha\) of exponent \(p\) with \(K \subset M \subset \overline{K}\). Let \(A = R' \cap M\) and let \(m = P' \cap A\).

Now \((A,m)\) is not a \(p\) \(N\)-pair since \(f(X)\) is a \(p\) \(N\)-polynomial over \((A,m)\) which does not have a root in \(m\). But \((A,m)\) is a \(k\) \(N\)-pair for \(2 \leq k < p\). For, let \(g(X)\) be a \((p - 1)N\)-polynomial over \((A,m)\). Then \(g(X)\) as an element of \(R'[X]\) has a unique root \(\beta \in P'\). Now \([M(\beta):M] \leq p - 1\), but by [7, Thm. 2], \([M(\beta):M] = p^i\) for some \(i \geq 0\). So \([M(\beta):M] = 1\) and \(\beta \in M\). Thus \(\beta \in m = P' \cap A\) and \((A,m)\) is a \((p - 1)N\)-pair. It follows by (2.6) that \((A,m)\) is a \(k\) \(N\)-pair for \(2 \leq k < p\).

**REMARK.** If \(j\) and the prime number \(p\) are closed such that \(p > \max \{C_{j,n} | n=0,1,\ldots,j\}\), then by Theorem 2.10, the above example is an example of a pair \((A,m)\) such that \((A,m)\) is not a \(p\) \(H\)-pair, but \((A,m)\) is a \(k\) \(H\)-pair for \(2 \leq k \leq j\).

Let the notation be as in the above example. Then \((A_m,mA_m)\) is as an example of a normal quasi-local domain which is not a \(p\) \(N\)-pair, but is a \(k\) \(N\)-pair for \(2 \leq k < p\).

6. **PROPERTIES OF \(k\) \(N\)-PAIRS.**

We conclude this paper by noting that many of the properties of the Henselization or \(N\)-closure of a pair which S. Greco proved in [5] also hold for a \(k\) \(N\)-closure and thus also for a \(2\) \(H\)-closure and a \(3\) \(H\)-closure. Some of these results are: direct limits commute with \(k\) \(N\)-closures, cf. [5, Cor. 3.6]; a \(k\) \(N\)-closure of \((A,m)\) is flat over \(A\) and is faithfully flat over \(A\) iff \(m \subset J(A)\), cf. [5, Thm. 6.5]; a \(k\) \(N\)-closure of a noetherian ring is noetherian, and if a \(k\) \(N\)-closure of \((A,m)\) is Noetherian and \(m \subset J(A)\), then \(A\) is Noetherian, cf. [5, Cor. 6.9]; if \(A\) is Noetherian
and A has one of the properties $R_k$, $S_k$, regular, or Cohen-Macaulay, then a $k$ $N$-closure of $(A, m)$ also has that property, and the converse is also true provided $m \subseteq J(A)$, cf. [5, Cor. 7.7]; a $k$ $N$-closure preserves locally normal, cf. [5, Thm. 9.7]; and a $k$ $N$-closure of a reduced ring is reduced, cf. [5, Thm. 8.7].

REFERENCES


