ON UNIFORM CONVERGENCE FOR
(\(\mu, \nu\))-TYPE RATIONAL APPROXIMANTS IN \(\mathbb{C}^n\) - II

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(Received June 3, 1980)

ABSTRACT. This paper shows that if \(f(z)\) is analytic in some neighborhood of the origin, but meromorphic in \(\mathbb{C}^n\) otherwise, with a denumerable non-accumulating pole sections in \(\mathbb{C}^n\), and if for each fixed \(\nu\), the pole set of each \((\mu, \nu)\)-unisolvent rational approximant \(\pi_{\mu\nu}(z)\) tends to infinity as \(\mu' = \min(\mu) + \infty\), then \(f(z)\) must be entire in \(\mathbb{C}^n\). This paper also shows a monotonicity property for the "error sequence" \(e_{\mu\nu} = ||f(z) - \pi_{\mu\nu}(z)||_K\) on compact subsets \(K\) of \(\mathbb{C}^n\).

KEY WORDS AND PHRASES. uniform convergence, entire functions, approximations and expansions.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 41.

1. INTRODUCTION.

Two earlier papers by Lutterodt [1,2] gave results on uniform convergence under restricted assumptions made about the \((\mu, \nu)\)-rational approximants. In [1], the \(n^1\)-type \((\mu, \nu)\)-rational approximants, were assumed to be uniformly bounded on a polydisk; whereas, in [2], the \((\mu,1)\)-rational approximants were under the assumption that the coefficients of the denominator polynomial of degree \(\nu = (1,1,\ldots,1) = 1\) vanished as \(\mu \to (\infty, \ldots, \infty)\) except for \(b_0^{(\mu)} \neq 0\). In fact, \(b_0^{(\mu)}\) is normalized to unity.

In this paper, we attempt to provide a general result about uniform convergence of \((\mu, \nu)\)-rational approximants to entire functions in \(\mathbb{C}^n\).
The main results of this paper are Theorems 1 and 2. Theorem 1 establishes uniform convergence for \((\mu, \nu)\) unisolvent rational approximants with infinite pole sections that tend to infinity as \(\mu \to (\infty, \ldots, \infty)\) on compact subsets of \(\mathbb{C}^n\); Theorem 2 introduces an "error sequence"

\[ e_{\mu \nu} = \|f(z) - \pi_{\mu \nu}(z)\|_K \]

on any compact subset of \(\mathbb{C}^n\) and shows that \(e_{\mu \nu}\) is monotonic in \(\nu\) for sufficiently large values of \(\mu\).

2. NOTATION AND DEFINITIONS.

Let \(z = (z_1, \ldots, z_n)\) be an \(n\)-tuple point in \(\mathbb{C}^n\); let \(\mu = (\mu_1, \ldots, \mu_n)\) and \(\nu = (\nu_1, \ldots, \nu_n)\) be \(n\)-tuples of non-negative integers in \(\mathbb{N}^n\).

Let \(\mathcal{R}_{\mu \nu}\) be the class of all rational functions of the form

\[ R_{\mu \nu}(z) = \frac{P_\mu(z)}{Q_\nu(z)}, \quad Q_\nu(0) \neq 0 \]

where \(P_\mu(z)\) and \(Q_\nu(z)\) are polynomials of multiple degree of at most \(\mu\) and \(\nu\), respectively, with \((P_\mu(z), Q_\nu(z)) = 1\) in some neighborhood of the origin.

**DEFINITION 1.** Suppose \(f(z)\) is analytic at the origin and \(f(0) \neq 0\). An \(R_{\mu \nu}(z) \in \mathcal{R}_{\mu \nu}\) is said to be a \((\mu, \nu)\)-type rational approximant to \(f(z)\) at \(z = 0\) if

\[ \frac{\partial |\lambda|}{\partial z^\lambda} (Q_\nu(z)f(z) - P_\mu(z))|_{z=0} = 0 \quad (2.1) \]

for \(\lambda \in E^{\mu \nu} \subset \mathbb{N}^n\), a lattice interpolation set with the following properties:

(i) \(0 \in E^{\mu \nu}\)
(ii) \(\lambda \in E^{\mu \nu} = \gamma \in E^{\mu \nu}, \gamma_i \leq \lambda_i \quad i = 1, \ldots, n\)
(iii) \(E_\mu = \{\lambda \in \mathbb{N}^n: 0 \leq \lambda_i \leq \mu_i, \quad i = 1, \ldots, n\} \subset E^{\mu \nu}\)
(iv) \(|E^{\mu \nu}| \leq \prod_{i=1}^{n-1} (\mu_i + 1) + \prod_{i=1}^{n} (\nu_i + 1) - 1\)
(v) Each projected variable has the Padé index set
(vi) Each \(\nu_i \leq \mu_i \quad i = 1, \ldots, n\).

Here \(|E^{\mu \nu}|\) is the cardinality of \(E^{\mu \nu}\) and

\[ \frac{\partial |\lambda|}{\partial z^\lambda} = \left( \frac{\lambda_1 + \ldots + \lambda_n}{\partial z_1 \ldots z_n} \right)^{\lambda_1 \ldots \lambda_n} \]
DEFINITION 2. An $R_{\mu\nu}(z) \in R_{\mu\nu}$ is said to have multiple degree $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ if, in the $z_j$-variable, $R_{\mu\nu}(z)$ expressed as a quotient of two pseudo-polynomials in $z_j$, has degree given by $\mu_j^* = \max(\mu_j, \nu_j)$, $1 \leq j \leq n$.

It follows from property (vi) of $E_{\mu\nu}$, that the multiple degree of a $(\mu, \nu)$-type rational approximant is always $\mu$.

We shall refer the reader to the definition of a unisolvent $(\mu, \nu)$-type rational approximant to $f(z)$ in Lutterodt [3]. We shall denote this by

$$\pi_{\mu\nu}(z) = \frac{\mu_{\nu}}{Q_{\mu\nu}(z)}.$$

We then normalize the denominator polynomial $Q_{\mu\nu}(z)$, dividing numerator and denominator by the modulus of largest coefficient of the denominator polynomial. Thus, we get

$$\pi_{\mu\nu}(z) = \frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}$$

where $Q_{\mu\nu}^*(z)$ is a normalized polynomial.

3. CONVERGENCE.

The uniform convergence for the $(\mu, \nu)$-rational approximants to $f(z)$ entire in $\mathbb{C}^n$ rests on the assumptions made about $f(z)$ and the hypothesis that, for each fixed multiple denominator degree $\nu$ of $\pi_{\mu\nu}(z)$, the pole set tends to infinity as $\mu \to (\infty, \ldots, \infty)$. In Theorem 1 below, we assume that $f(z)$ is possibly meromorphic, not with a finite pole set as in Theorem 2 of [3], but with a pole set having infinite sections such that only a finite number of such pole sections overlap with any given polydisk. Thus, Theorem 1 of this paper extends the result in [3].

THEOREM 1: Suppose $f(z)$ is analytic at the origin and is possibly meromorphic with an infinite pole set in $\mathbb{C}^n$ without accumulation of pole sections such that given $\rho > 1$, the polydisk

$$A_\rho^n = \{ z \in \mathbb{C}^n : |z_j| < \rho, \ j = 1, \ldots, n, \ \rho > 1 \}$$

overlaps with only a finite number of these pole sections.

Suppose $\pi_{\mu\nu}(z)$ is a unisolvent $(\mu, \nu)$-rational approximant to $f(z)$ such that for each fixed $\nu$, the pole set of $\pi_{\mu\nu}(z)$ tends to infinity as $\mu \to (\infty, \ldots, \infty)$. Then

(i) $f(z)$ must be entire in $\mathbb{C}^n$

(ii) $\pi_{\mu\nu}(z) \to f(z)$ uniformly on every compact subset of $\mathbb{C}^n$. 
THEOREM 2: Suppose the conditions of Theorem 1 are satisfied. Let $K$ be any compact subset of $\mathbb{C}^n$. Let

$$e_{\mu \nu} = \| f(z) - \pi_{\mu \nu}(z) \|_K = \sup_{z \in K} |f(z) - \pi_{\mu \nu}(z)|$$

for each fixed $\nu$.

Then for sufficiently large $\nu$, $e_{\mu \nu}$ is monotonic in $\nu$ and satisfies

$$e_{\mu, \nu+1} \leq e_{\mu \nu} \quad \text{with} \quad \nu_j \leq \nu_j + 1, \quad 1 \leq j \leq n.$$

**Lemma 1.** Let $\nu$ be fixed and let $Q_{\mu \nu}^* (z)$ be a normalized denominator polynomial of $\pi_{\mu \nu}(z)$. The zero set of $Q_{\mu \nu}^* (z)$ tends to infinity as $\mu \to (\infty, \ldots, \infty) = Q_{\mu \nu}^* (z)$ tends to a constant.

**Proof.** Suppose the result is false; i.e., for fixed $\nu$, $Q_{\mu \nu}^{-1} (0)$ tends to infinity, but $Q_{\mu \nu}^* (z)$ does not tend to a constant.

By Lemma 1 in [3], given $\rho > 1$ and a polydisk $\Delta^n_\rho$, and $\mu$ sufficiently large,

$$Q_{\mu \nu}^{-1} (0) \cap \Delta^n_\rho = \emptyset$$

Suppose that $Q_{\mu \nu}^* (z) + Q_m^* (z)$ is not constant as $\mu \to (\infty, \ldots, \infty)$ where $m = (m_1, \ldots, m_n)$ s.t. $m_i \leq \nu_i$, $1 \leq i \leq n$ and that $Q_m^* (z)$ is a polynomial of multiple degree in less than $\nu$ in a partial ordered sense. Then since $Q_m^* (z)$ is non-constant, it has a set of non zero coefficients. Thus, $Q_m^{-1} (0)$, the zero set of $Q_m^* (z)$ cannot be empty. Now, taking $\rho_0 > 1$, we find that

$$Q_m^{-1} (0) \cap \Delta^n_{\rho_0} \neq \emptyset$$

a contradiction. Hence the above supposition must be false and the Lemma holds.

**Proof of Theorem 1.** $f(z)$ is analytic at $z = 0$ and is possibly meromorphic with an infinite pole set

$$G = \bigcup_{k=1}^{\infty} G_{\rho_k}$$

where

$$G_{\rho_k} \subset G_{\rho_{k+1}}$$

and
$G_{\sigma_k} := \{z \in \mathbb{C}^n : q_{\sigma_k}(z) = 0\}$.

$q_{\sigma_k}(z)$ is a polynomial of at most multiple degree,

$$\sigma_k = (\sigma_{k1}, \ldots, \sigma_{kn}).$$

Given any real number $\rho > 1$, and a polydisk $\Delta^n_{\rho}$, then $k_o = k_o(\rho)$ such that the zero set $G_{\sigma_{k_o}}$ overlaps the polydisk $\Delta^n_{\rho}$. Now, by Theorem 1 of [3], if we choose $\nu = \sigma_{k_o}$, then we must have on $\Delta^n_{\rho}$ as $\mu \to (\infty, \ldots, \infty)$

$$\Delta^n_{\rho} \cap Q_{\mu\nu}^{-1}(0) = \Delta^n_{\rho} \cap G_{\sigma_{k_o}}$$

But by hypothesis, the pole set of $\pi_{\mu\nu}(z)$ tends to infinity as $\mu \to (\infty, \ldots, \infty)$ for each fixed $\nu$. Therefore, for the given $\rho > 1$ above as $\mu \to (\infty, \ldots, \infty)$, we must have

$$\Delta^n_{\rho} \cap Q_{\mu\nu}^{-1}(0) = \emptyset$$

Thus by (3.4) and (3.5) we must have

$$\Delta^n_{\rho} \cap G_{\sigma_{k_o}} = \emptyset.$$

Since $k_o = k_o(\rho)$ and $\rho$ is arbitrary, it follows that $G_{\sigma_{k_o}}$ must tend to infinity as $k_o \to \infty$. Hence, all the poles of $f(z)$ must tend to infinity and $f(z)$ must therefore be entire. This completes (i).

To prove (ii), we note that the result follows immediately from Theorem 1 of [3] and the (i) part just proved above.

**Proof of Theorem 2.** Let $K$ be any compact subset of $\mathbb{C}^n$. Then we can find $\rho > 1$ and a polydisk $\mathbb{C}^n$ such that $K \subset \Delta^n_{\rho}$. Then, for $\mu$ sufficiently large and $z \in K$, we find by the hypothesis of Theorem 1, that for each fixed $\nu$,

$$Q_{\mu\nu}^*(z) \neq 0 \quad \text{i.e.} \quad \delta > 0$$

such that

$$|Q_{\mu\nu}^*(z)| > \delta.$$  

Hence, under these conditions, we get

$$||\pi_{\mu, \nu+1}(z) - \pi_{\mu\nu}(z)||_K \leq \frac{2||P_{\mu\nu}(z)||_K}{\delta^2} ||Q_{\mu, \nu+1}(z) - Q_{\mu\nu}(z)||_K.$$
By Lemma 1, we know that $\mathcal{Q}_{\mu\nu}^*(z)$ tends to a constant as $\mu \to (\infty, \ldots, \infty)$ for any fixed $\nu$. Hence, given $\varepsilon > 0$, $\varrho_0 = (\mu_{10}, \ldots, \mu_{n0})$ such that for $\mu_{10} < \mu_i$, $1 \leq i \leq n$

\[ \| \mathcal{Q}_{\mu,\nu+1}^*(z) - \mathcal{Q}_{\mu\nu}^*(z) \|_K < \varepsilon \frac{\delta^2}{2M}. \tag{3.7} \]

$M_\rho = \| \mathcal{P}_{\mu\nu}^*(z) \|_{\Delta_\rho} \geq \| \mathcal{P}_{\mu\nu}^*(z) \|_K$ by the maximum modulus principle, and $M_\rho$ is dependent on $\rho$ but independent of $\mu$. Hence, by combining (3.6), (3.7) and (3.8) for each fixed $\nu$ and $\mu_{10} < \mu_i$, $1 \leq i \leq n$, we obtain

\[ \| \pi_{\mu,\nu+1}(z) - \pi_{\mu\nu}(z) \|_K < \varepsilon. \tag{3.8} \]

To get the desired inequality, we note by triangular for sup-norms on $K$ that

\[ e_{\mu,\nu+1} \leq e_{\mu\nu} + \| \pi_{\mu,\nu+1}(z) - \pi_{\mu\nu}(z) \|_K, \tag{3.9} \]

where we have used the definition of $e_{\mu\nu}$ as in (3.1).

For $\mu_{10} < \mu_i$, $1 \leq i \leq n$, and for each fixed $\nu$,

\[ e_{\mu,\nu+1} < e_{\mu\nu} + \varepsilon \]

Since $\varepsilon > 0$ is arbitrary, the results follows.

ACKNOWLEDGEMENT. This paper was written while I was at the Mathematics Department, University of South Florida, Tampa, Florida.

REFERENCES

