ON LINEAR ALGEBRAIC SEMIGROUPS III

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ABSTRACT. Using some results on linear algebraic groups, we show that every connected linear algebraic semigroup $S$ contains a closed, connected diagonalizable subsemigroup $T$ with zero such that $E(T)$ intersects each regular $J$-class of $S$. It is also shown that the lattice $(E(T), <)$ is isomorphic to the lattice of faces of a rational polytope in some $\mathbb{R}^n$. Using these results, it is shown that if $S$ is any connected semigroup with lattice of regular $J$-classes $U(S)$, then all maximal chains in $U(S)$ have the same length.

KEY WORDS AND PHRASES. Linear algebraic semigroup, idempotent, polytope.

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0. INTRODUCTION.

Throughout this paper, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{Z}^+$, $\mathbb{Q}^+$ will denote the sets of reals, integers, rationals, positive reals, positive integers and positive rationals respectively. If $X$ is a set then $|X|$ denotes the cardinality of $X$. If $X$ is a subset of a semigroup, then $<X>$ denotes the subsemigroup generated by $X$. If $(P, \leq)$ is a partially ordered set and $\{a_1 < a_2 < \ldots < a_n\}$ is a finite chain in $P$, then we define the length of the chain to be $n - 1$. $K$ will denote a fixed algebraically closed field, $K^n = K \times \ldots \times K$ the affine $n$-space. $M_n(K)$ will denote the set of all $n \times n$ matrices and $GL(n, K)$ the group of units of $M_n(K)$. $F(x_1, \ldots, x_n)$ will denote the free commutative semigroup in the variables $x_1, \ldots, x_n$ and $K[x_1, \ldots, x_n]$
the free commutative algebra over $K$ in the variables $X_1, \ldots, X_n$. We use the notation of [6,7] for algebraic semigroups. Let $S$ be an algebraic monoid with identity element 1 and group of units $G$. If $g \in G$, then the maps $x \mapsto xg$, $x \mapsto gx$, $x \mapsto g^{-1}xg$ are all automorphisms of the variety $S$. The last one is also a semigroup automorphism. If we let $\tilde{G} = \{(a,b) : a,b \in S, ab = 1\}$, then $\tilde{G}$ becomes an algebraic group. Actually, with more general notions of varieties [5], $G$ itself can be viewed as an algebraic group. By [6, Theorem 1.1], we can assume that $S$ is a closed submonoid of some $M_n(K)$. Then clearly $G = GL(n,K) \cap S$ and $S \setminus G$ is closed.

If $S_1$ is a closed submonoid of $S$ with group of units $G_1$, then $G_1 = G \cap S_1$. If $H$ is a closed subgroup of $G$, then $H$ is the group of units of $\overline{H}$. If $S$ is connected, then clearly so is $G$, $\overline{G} = S$ and $\dim S = \dim G$. If $S$ is not connected, then [7; Lemma 1.9], 1 lies in a unique irreducible component $S_1$ of $S$ and $S_1$ is a closed connected submonoid of $S$. We say that $S$ is trigonalizable if $S$ is $\ast$-isomorphic to a closed semigroup of lower triangular matrices. If $S$ is connected, then since $\overline{G} = S$, it follows from the Lie-Kolchin Theorem [5; Theorem 17.6] that $S$ is trigonalizable if and only if $G$ is solvable. $S$ is a $d$-semigroup if $S$ is $\ast$-isomorphic to a closed subsemigroup of $(K^P, \cdot)$ for some $p \in \mathbb{Z}^+$. If $S$ is connected, then since $\overline{G} = S$, we see that $S$ is a $d$-semigroup if and only if $G$ is a torus. By [7; Corollary 3.15], a connected $d$-semigroup with zero can be characterized as a connected Clifford semigroup with zero. If $X,Y \subseteq S$, then $X$ is conjugate to $Y$ if $g^{-1}Xg = Y$ for some $g \in G$.

1. CONNECTED SEMIGROUPS

**Lemma 1.1.** Let $S$ be a connected monoid, $e \in E(S)$, $e \neq 1$. Then there exists a closed connected submonoid $S'$ of $S$ such that $1, e \in S'$ and $e$ is the zero of $S'$.

**Proof.** Let $G$ denote the group of units of $S$ and set $V = S \setminus G$. Then $V = V_1 \cup \ldots \cup V_r$ where $V_1, \ldots, V_r$ are closed and irreducible. Let $m_i = \dim V_i$, $i = 1, \ldots, r$. Then $m_i \leq n-1$, where $n = \dim S$. Let $\phi : S + eS$ be given by $\phi(x) = ex$. Let $q = \dim eS$, $\phi_i = \phi|_{V_i}$, $W_i = \overline{\phi(V_i)} \subseteq eS$. Let $i \in \{1, \ldots, r\}$. Suppose $W_i = eS$.

Then $\phi_i$ is dominant. So by [5; Theorem 4.3], there exists a non-empty open set $O_i$ of $eS$ such that $0 \subseteq \phi_i(V_i)$ and so that $\dim \phi_i^{-1}(x) = m_i - q < n - q$ for all $x \in O_i$. So
Next suppose $\mathcal{W}_i \neq \mathcal{E}$. Then set $0_i = \mathcal{E} \setminus \mathcal{W}_i$. Then
\[ V_i \cap \phi^{-1}(x) = \emptyset \text{ for all } x \in 0_i. \] (2)

Let $0 = 0_1 \cap 0_2 \cap \ldots \cap 0_r$. Since $\mathcal{E}$ is connected, $0 \neq \emptyset$. Let $x \in 0$. Then $x \in \phi^{-1}(x)$. Let $D$ be an irreducible component of $\phi^{-1}(x)$ such that $x \in D$. Then $\dim D > n - q$. We claim that $D \not\subseteq V$. For suppose $D \subseteq V$. Then $D \subseteq V_i$ for some $i$. Since $x \in \phi^{-1}(x) \cap 0 \cap V_i$, (2) is ruled out. So by (1) $\dim D < n - q$, a contradiction. Hence $D \not\subseteq V$. So $D \cap G \neq \emptyset$. Let $g \in D \cap G$. So $\phi(g) = x$. Thus $eg = x$, $xg^{-1} = e$. Let $Y = Dg^{-1}$. Then $Y$ is closed and irreducible.

Let $y \in Y$. Then $yg = x$ and $ey = xg^{-1} = e$. Hence $ey = e$ for all $y \in Y$. Since $g \in D$, $1 = gg^{-1} \in Y$. Since $x \in D$, $e = xg^{-1} \in Y$. Let $S_1 = \{a \mid a \in S, ea = e\}$. Then $S_1$ is a closed submonoid of $S$ and $Y \subseteq S_1$. Let $S_2$ be the (unique) irreducible component of $1$ in $S_1$. Then $Y \subseteq S_2$ and $S_2$ is a closed connected submonoid of $S$. Thus $1$, $e \in S_2$ and $ea = e$ for all $a \in S_2$. By the dual of the above argument, there exists a closed connected submonoid $S_3$ of $S_2$ such that $e \in S_3$ and $ae = e$ for all $a \in S_3$. So $ae = ea = e$ for all $a \in S_3$.

**FACT 1.2.** Let $A \subseteq M_n(K)$ such that $AB = BA$ for all $A, B \in A$. Suppose also that each $A \in A$ is lower triangular and diagonalizable. Then there exists a lower triangular, invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

**PROOF.** We prove by induction on $n$. Let $A = \{A_\alpha \mid \alpha \in \Omega\}$, $A_\alpha = \begin{bmatrix} a & 0 \\ \alpha & b \alpha \end{bmatrix}$. $C_\alpha$ is $(n-1) \times (n-1)$, $a_\alpha \in K$. Clearly $C_\alpha C_\beta = C_\beta C_\alpha$ for all $\alpha, \beta$. Since

\[ \text{minimum polynomial of } C_\alpha = \text{minimum polynomial of } A_\alpha, \]

minimum polynomial of $C_\alpha$ has no multiple roots. So each $C_\alpha$ is diagonalizable. So there exists, by induction, an invertible, lower triangular $(n-1) \times (n-1)$ matrix $M_\alpha$ such that $M_\alpha^{-1}C_\alpha M_\alpha$ is diagonal for all $\alpha$. Let $M = \begin{bmatrix} 1 & 0 \\ 0 & M_1 \end{bmatrix}$. Then
\[ D_\alpha = M^{-1}_\alpha A M = \begin{bmatrix} a_\alpha & 0 \\ F & G \alpha \end{bmatrix}, \quad G_\alpha \text{ is (n-1) x (n-1) and diagonal.} \]

Let \( E_\alpha = D_\alpha - a_\alpha I, \quad \alpha \in \Omega. \) Then each \( E_\alpha \) is diagonalizable and \( E_\alpha E_\beta = E_\beta E_\alpha \) (all \( \alpha, \beta \)). Moreover,

\[
E_\alpha = \begin{bmatrix}
0 & 0 \\
0 & b_2(\alpha) & c_2(\alpha) \\
& \ddots & \ddots \\
& & 0 & b_n(\alpha) & c_n(\alpha)
\end{bmatrix}
\]

Since \( E_\alpha E_\beta = E_\beta E_\alpha \),

\[ c_i(\alpha) b_i(\beta) = b_i(\alpha) c_i(\beta), \quad i = 2, \ldots, n, \text{all } \alpha, \beta \quad (3) \]

Also, since \( E_\alpha \) is diagonalizable,

\[ c_i(\alpha) = 0 \text{ implies } b_i(\alpha) = 0, \text{ all } i, \alpha \quad (4) \]

Let \( i \in \{2, \ldots, n\} \). If there exists \( \alpha \) such that \( c_i(\alpha) \neq 0 \), let \( u_1 = -b_i(\alpha)/c_i(\alpha) \).

By (3), \( u_1 \) is independent of the choice of \( \alpha \). If there is no such \( \alpha \), let \( u_1 = 0 \).

Let \( u_1 = 1 \) and set \( u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \). By (4), \( E_\alpha u = 0 \) for all \( \alpha \). Let \( e_i \) be the column with 1 in \( i \)th component and 0 elsewhere. Then \( u, e_2, \ldots, e_n \) is a linearly independent set of eigenvectors of \( E_\alpha \) for all \( \alpha \in \Omega \). Let \( R = [u, e_2, \ldots, e_n] \). Then \( R \) is lower triangular and invertible. Clearly \( R^{-1} E_\alpha R \) is diagonal for all \( \alpha \). So \( R^{-1} D_\alpha R \) is diagonal for all \( \alpha \). Let \( P = MR \).

**Lemma 1.3.** Let \( S \) be a connected monoid with identity element \( l \), zero \( e \).

Let \( G \) denote the group of units of \( S \). Suppose \( G \) is solvable. Then for any maximal torus \( T \) of \( G \), \( e \in T \).

**Proof.** We can assume that \( S \) is a closed submonoid of \( M_n(K) \). By the Lie-Kolchin theorem [5; Theorem 17.6] there exists \( P \in GL(n,K) \) such that \( P^{-1}GP \) is lower triangular. Since \( G = S \), \( P^{-1}SP \) is lower triangular. So we can assume
that $S$ is lower triangular. Let $T$ be a maximal torus of $G$ and set $X = T \cup \{e\}$. Then $X$ satisfies the hypothesis of Fact 1.2. So there exists a lower triangular $R \in \text{GL}(n,K)$ such that $R^{-1}XR$ is diagonal. Clearly $R^{-1}SR$ remains lower triangular. So we can assume that $X$ is diagonal. If $a \in S$, then let $\phi(a)$ be the $n \times n$ diagonal matrix, with the diagonal being that of $a$. Then $\phi(X) = X$. Clearly $\phi$ is a $^*$-homomorphism of $S$ into $M_n(K)$ and $\phi(G)$ is a torus. By [5; Corollary 21.3C], $\phi(G) = \phi(T) = T$. So $\phi(G) \subseteq S$. Since $G = S$, $\phi(S) \subseteq S$. Let $W = \phi(S)$. Then $W = \{a| a \in S, \phi(a) = a\}$ is closed. Since $\phi(S) = W$, $W$ is a closed connected submonoid of $S$. Let $H$ denote the group of units of $W$. Then $T \subseteq H \subseteq G$ and $H$ is a torus. So $T = H$ and $W = T$. Clearly $e = \phi(e) \in W = T$.

**THEOREM 1.4.** Let $S$ be a connected monoid with group of units $G$. Let $B$ be a Borel subgroup of $G$. Then $S \cup xBx^{-1}$.

**PROOF.** We can assume that $S$ is a closed submonoid of $W = M_n(K)$. Let $G_1 = \{(a,a^{-1})| a \in G\}$ then $G_1$ is a closed subset of $W \times W$. If $(a,b),(c,d) \in G_1$, then define $(a,b)(c,d) = (ac,db)$. Then $G_1$ is an algebraic group $^*$-isomorphic to $G$. Let $B_1 = \{(a,a^{-1})| a \in B\}$. Then $B_1$ is a Borel subgroup of $G_1$. Now [5; Theorem 21.3], $G_1/B_1$ is a projective variety. Let $\phi: G_1 \to G_1/B_1$ be the natural projection $\phi(a) = aB_1$. Let $V = W \times G_1, Y = W \times G_1/B_1$. By [1; Theorem 6.8], $G_1/B_1$ is smooth and hence a normal variety. The same is true for $W$. So $Y = W \times G_1/B_1$ is normal [1; p. 77]. Let $\psi: V \to Y$ be given by $\psi(a,b) = (a,\phi(b))$. Then $\psi$ is a surjective morphism. Clearly each fibre of $\psi$ is irreducible and has dimension equal to that of $B_1$. So [1; Proposition 18.4], $\psi$ is an open map. Let $X = \{(a,g,g^{-1})| a \in S, g \in G, g^{-1}ag \in B\}$. Then $X$ is closed in $V$. So $\psi(\psi(X))$ is open in $Y$. Hence $\psi(\psi(X))$ is closed in $Y$. Clearly $\psi(\psi(X)) \subseteq \psi(X)$. Suppose $\omega \in \psi(X), \omega \in \psi(\psi(X))$. Then $\omega = \psi(x) = \psi(y)$ for some $x \in X, y \in \psi(X)$. So $x = (a,g,g^{-1}), y = (a,h,h^{-1})$ for some $a \in S, g, h \in G$. Now $g^{-1}ag \in B$. Since $\psi(x) = \psi(y), \phi(g,g^{-1}) = \phi(h,h^{-1})$. So $gB = hB$. Thus $h = gb$ for some $b \in B$. So $h^{-1}ah = b^{-1}(g^{-1}ag)b \in b^{-1}Eb = B$, a contradiction. So $\psi(\psi(X)) = \psi(X)$ and $\psi(X)$ is closed. Let $\theta: Y = W \times G_1/B_1 \to W$ denote the projection of $Y$ onto $W$. Then since $G_1/B_1$ is projective, $\theta$ is a closed
map [5; Theorem 6.2]. Hence $\theta(\psi(x))$ is closed in $W$. Clearly $\theta(\psi(x)) = \bigcup_{g \in S} g\overline{e(x)}^{-1} \subseteq S$. By [5; Theorem 22.2], $G \subseteq \theta(\psi(x))$. Since $\overline{G} = S$, $\theta(\psi(x)) = S$. This proves the theorem.

**COROLLARY 1.5.** Let $S$ be a connected monoid with zero $e$ and $T$ a maximal torus in the group of units $G$ of $S$. Then $e \in \overline{T}$.

**PROOF.** Now $T \subseteq B$ for some Borel subgroup $B$ of $G$. By Theorem 1.4, $e \in xBx^{-1}$ for some $x \in G$. So $e = x^{-1}ex \in B$. Hence $e$ is the zero of $B$. By Lemma 1.3, $e \in \overline{T}$.

**COROLLARY 1.6.** Let $S$ be a connected monoid with group of units $G$. Let $e_1, \ldots, e_k \in E(S)$ such that $e_1 > e_2 > \ldots > e_k$. Then there exists a maximal torus $T$ of $G$ such that $e_1, \ldots, e_k \in \overline{T}$.

**PROOF.** We prove by induction on $k$. If $k = 1$, we are done by Lemma 1.1 and Corollary 1.5. So assume $k > 1$. By Lemma 1.1, there exist closed connected submonoids $S_1, \ldots, S_k$ of $S$ such that $e_i$ is the zero of $S_i$. Then $ae_k = e_k a = e_k$ for all $a \in S_i$, $i = 1, \ldots, k$. Let $V = \{a | a \in S, ae_k = e_k a = e_k\}$. Then $V$ is a closed submonoid of $S$ and $S_1, \ldots, S_k \subseteq V$. Let $W$ be the (unique) irreducible component of $1$ in $V$. Then $S_1, \ldots, S_k \subseteq W$ and $W$ is a closed connected submonoid of $S$. So $e_k$ is the zero of $S$. Let $G_1$ denote the group of units of $W$. By our induction hypothesis, there exists a maximal torus $T_1$ and $G_1$ such that $e_1, \ldots, e_{k-1} \in \overline{T_1}$. By Corollary 1.5, $e_k \in \overline{T_1}$. Let $T_1 \subseteq T$ where $T$ is a maximal torus of $G$. Then $e_1, \ldots, e_k \in \overline{T}$.

By [7; Lemma 1.3] we have,

**LEMMA 1.7.** Let $S$ be a semigroup, $J_1, \ldots, J_k \in U(S)$, $J_1 > J_2 > \ldots > J_k$. Then there exists $e_1, \ldots, e_k \in E(S)$ such that $e_i \in J_i$, $i = 1, \ldots, k$ and $e_1 > e_2 > \ldots > e_k$.

**THEOREM 1.8.** Let $S$ be a connected monoid with group of units $G$. Then

1. All maximal closed connected d-submonoids of $S$ are conjugate.
2. All maximal closed connected d-submonoids with zeroes, of $S$, are conjugate.
3. Let $Y$ be a maximal closed connected d-submonoid with zero, of $S$. Then $\bigcup_{g \in G} gE(Y)g^{-1} = E(S)$. In particular $E(Y) \cap J \neq \emptyset$ for all $J \in U(S)$.
PROOF. Since the group of units of a maximal closed connected d-submonoid of $S$ is a maximal torus in $G$, (1) follows from [5; Corollary 21.3A].

(2) Let $S_1$, $S_2$ be two maximal closed connected d-submonoids with zeroes of $S$. Let $e_i$ be the zero of $S_i$, $i = 1, 2$. Let $H_i$ be the group of units of $S_i$, $i = 1, 2$. Then $H_i \subseteq T_i$, $T_i$ a maximal torus of $G$, $i = 1, 2$. Let $V_i = \overline{T_i}$, $i = 1, 2$. Let $f_i$ be the minimum idempotent of $V_i$. Then $e_i \geq f_i$. Let $W_i = \{a | a \in V_i, af_i = f_i\}$, $U_i$ the (unique) irreducible component of 1 in $W_i$. Since $V_1$, $V_2$ are conjugate by (1), so are $W_1$, $W_2$. Hence $U_1$, $U_2$ are conjugate. Since $S_1 \subseteq W_i$, $S_1 \subseteq U_i$, $i = 1, 2$. By Lemma 1.1, $f_i \in U_i$, $i = 1, 2$. By the maximality of $S_i$, $S_i = U_i$, $i = 1, 2$.

(3) Let $e \in E(S)$. By Corollary 1.6, $e \in S_1$ for some closed connected d-submonoid $S_1$ of $S$. By [7; Theorem 3.16], there exists a closed connected d-submonoid with zero, $S_2$ of $S_1$ such that $e \in S_2$. By (2) $xs_2x^{-1} \subseteq Y$ for some $x \in G$. So $xex^{-1} \subseteq Y$. Hence $E(Y) \cap J_e \neq \emptyset$. Next let $J_1 > J_2 > \ldots > J_k$ be a maximal chain in $U(S)$. By Lemma 1.7, there exist $e_i \in E(J_i)$ such that $e_1 > e_2 > \ldots > e_k$. Clearly

$$e_1 > e_2 > \ldots > e_k$$

is a maximal chain in $E(S)$. For if $e_i > f > e_{i+1}$, $f \in E(S)$, then $J_e > J_f > J_{e_{i+1}}$, a contradiction. By Corollary 1.6, $e_1, \ldots, e_k \in M_1$ for some closed connected d-submonoid $M_1$ of $S$. By [7; Theorem 3.16], $e_1, \ldots, e_k \in M_2$ for some closed connected d-submonoid with zero, $M_2$ of $M_1$. So $e_1, \ldots, e_k \in M_3$ for some maximal connected d-submonoid with zero, $M_3$ of $S$. Since (5) is maximal in $E(S)$, it is maximal in $E(M_3)$. By [7; Theorem 3.17], $\dim M_3 = k - 1$. By (2) $\dim M_3 = \dim Y$.

THEOREM 1.9. Let $S$ be a connected semigroup. Then all maximal chains in $U(S)$ have the same length.

PROOF. $U(S)$ has maximum element $J_0$. Let $e \in E(J_0)$. By [7; Lemma 1.3, 1.7]. $U(eS) = \{J \cap eS \mid J \in U(S)\} \subseteq U(S)$. Now $eS$ is a connected monoid. We are done by Theorem 1.8 (3).

THEOREM 1.10. Let $S$ be a connected monoid such that for all $a, b \in S$, $a \mid b$
implies \(a^2|b^i\) for some \(i \in \mathbb{Z}^+\). Let \(Y\) be a maximal closed connected \(d\)-submonoid with zero, of \(S\). Then \(J \cap Y\) is a subgroup of \(Y\) for all \(J \in U(S)\). In particular \(U(Y) = \{J \cap Y | J \in U(S)\}\) and \((U(S), \leq) \cong (E(Y), \leq)\).

**PROOF.** The hypothesis implies by [8] that \(J\) is a subsemigroup of \(S\) for all \(J \in U(S)\). Let \(J \in U(S)\). Then \(J \cap Y \neq \emptyset\) by Theorem 1.8. Let \(a, b \in J \cap Y\). Then \(ae, bh \in Y\) for some \(e, f \in E(Y)\). So \(e, f \in J\). Since \(e, f \in Y\), \(ef = fe\).

Since \(J\) is completely simple, \(e = f\). So \(ahb \in Y\) and \(J \cap Y\) is a subgroup of \(Y\).

Now applying the proof of Theorem 1.9 we have,

**COROLLARY 1.11.** Let \(S\) be a connected semigroup such that for all \(a, b \in S\),
\[a|b\] implies \(a^2|b^i\) for some \(i \in \mathbb{Z}^+\). Then \((U(S), \leq) \cong (E(Y), \leq)\) for some connected \(d\)-monoid with zero, \(Y\).

**THEOREM 1.12.** Let \(S\) be a connected monoid such that the group of units \(G\) of \(S\) is nilpotent. Then \(E(S)\) is finite.

**PROOF.** By [5; Proposition 19.2], \(G\) has a unique maximal torus \(T\). So \(\overline{T}\) is the unique maximal closed connected \(d\)-submonoid of \(S\). By Theorem 1.8, \(E(S) \subseteq \overline{T}\). So \(E(S)\) is finite.

**EXAMPLE 1.13.** \(S = \left\{\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in K\right\}\) is an example of a connected monoid satisfying the hypothesis of Theorem 1.12.

**CONJECTURE 1.14.** Let \(S\) be a connected monoid with zero such that \(E(S)\) is finite. Then the group of units of \(S\) is solvable.

**EXAMPLE 1.15.** Let \(S = \left\{\begin{bmatrix} a & b \\ 0 & a^2 \end{bmatrix} \mid a, b \in S\right\}\). Then \(S\) is a connected monoid with zero and \(|E(S)| = 2\). The group of units of \(S\) is solvable but not nilpotent.

2. **CONNECTED \(d\)-SEMIGROUPS WITH ZEROS**

Let \(S\) be a connected \(d\)-semigroup with zero, \(\dim S > 0\). Then \(S\) is a monoid [7; Theorem 2.7]. By a character of \(S\), we mean a \(\ast\)-homomorphism \(\chi : S \to K\) such that \(\chi(1) = 1, \chi(0) = 0\). We let \(\Phi(S)\) denote the commutative semigroup of all characters of \(S\) with pointwise multiplication. It is clear that if \(S_1, S_2\) are connected \(d\)-
semigroups with zeros, \( \dim S > 0 \), then \( S \cong -isomorphic to \( S_2 \) implies \( \phi(S_1) \cong \phi(S_2) \).

A commutative semigroup \( W \) is said to be totally cancellative if \( W \) is cancellative and for all \( a, b \in W, n \in \mathbb{Z}^+ \), \( a^n = b^n \) implies \( a = b \). We will need the following result of Grillet [3; Theorem 2.2].

**Theorem A** [Grillet]. Let \( W \) be a finitely generated commutative semigroup. Then \( W \) can be embedded in a free commutative semigroup if and only if \( W \) is totally cancellative and idempotent-free.

**Lemma 2.1.** Let \( S \) be a connected \( d \)-semigroup with zero \( 0 \) and identity \( 1 \), \( \dim S > 0 \). Then

1. \( \phi(S) \neq \emptyset \).
2. If \( e \in E(S), e \neq 0 \), then there exists \( \chi \in \phi(S) \) such that for all \( g \in E(S), g \leq e \) implies \( \chi(g) = 1 \), \( g \neq e \) implies \( \chi(g) = 0 \).
3. \( \phi(S) \) is idempotent-free and totally cancellative.
4. \( \phi(S) \) is linearly independent in the vector space of all functions from \( S \) into \( K \).

**Proof.** Let \( G \) denote the group of units of \( S \). We can assume that \( S \) is a closed submonoid of \( M_n(K) \) for some \( n \in \mathbb{Z}^+ \). If \( a \in S \), let \( \alpha(a) = \det a \). Then \( \alpha \in \phi(S), \alpha(f) = 0 \) for \( f \in E(S), f \neq 1 \). So \( \phi(S) \neq \emptyset \). Let \( e \in E(S), e \neq 0 \).

Then by the above, there exists \( \beta \in \phi(eS) \) such that \( \beta(f) = 0 \) for all \( f \in E(eS) \) with \( f \neq e \). Define \( \chi : S \to K \) as \( \chi(a) = \beta(\alpha(a)) \). This proves (2).

Let \( \chi \in \phi(S) \) such that \( \chi^2 = \chi \). Then \( \chi(S) = \{1, 0\} \) contradicting the fact that \( S \) is connected. So \( \phi(S) \) is idempotent-free. Let \( \chi_1, \chi_2, \chi_3 \in \phi(S) \) such that \( \chi_1 \chi_2 = \chi_1 \chi_3 \). If \( a \in G \), then \( \chi_1(a) \neq 0 \) and so \( \chi_2(a) = \chi_3(a) \). So \( \chi_2 = \chi_3 \) on \( G \). Since \( \overline{G} = S \), \( \chi_2 = \chi_3 \) on \( S \). So \( \phi(S) \) is cancellative. Now let \( \chi_1, \chi_2 \in \phi(S), m \in \mathbb{Z}^+ \) such that \( \chi_1^m = \chi_2^m \). Let \( Y = \{\xi | \xi \in K, \xi^m = 1\} \). Then \( Y \) is finite. If \( \xi \in Y \), let \( S_\xi = \{a | a \in S, \chi_1(a) = \xi \chi_2(a)\} \). If \( a \in G \), then \( \chi_1(a) \neq 0 \) for \( i = 1, 2 \).

So \( a \in S_\xi \) for some \( \xi \in Y \). Thus \( \overline{G} \subseteq \bigcup_{\xi \in Y} S_\xi \). Since \( \overline{G} = S \), \( S = \bigcup_{\xi \in Y} S_\xi \). Since \( S \) is connected, \( S = S_\xi \) for some \( \xi \in Y \). In particular, \( 1 = \chi_1(1) = \xi \chi_2(1) = \xi \). So \( \chi_1 = \chi_2 \) and \( \phi(S) \) is totally cancellative. This proves (3).
Now let $\chi_1, \ldots, \chi_m \in \Phi(S)$ be distinct characters of $S$ which are linearly dependent. Let $\psi_i$ denote the restriction of $\chi_i$ to $G$. Then $\psi_i, i = 1, \ldots, m$ are linearly dependent homomorphism of $G$. So [5, Lemma 16.1], $\psi_i = \psi_j$ for some $i \neq j$. Since $G = S$, $\chi_i = \chi_j$, contradiction. This proves the lemma.

**Lemma 2.2.** Let $S$ be a connected d-semigroup with zero, $\dim S > 0$. Then

1. $S$ is $\ast$-isomorphic to a closed submonoid $S'$ of $(K^n, \cdot)$ for some $n \in \mathbb{Z}^+$ such that $0 = (0, \ldots, 0) \in S'$.

2. $\Phi(S)$ is finitely generated.

3. If $a, b \in S$, $s \neq b$, then there exists $\chi \in \Phi(S)$ such that $\chi(a) \neq \chi(b)$.

**Proof.** First we prove (1). We can assume that $S$ is closed subsemigroup of $(K^n, \cdot)$ for some $n \in \mathbb{Z}^+$, $n$ minimal. Let $e$ denote the zero of $S$ and set $S_1 = \{a - e | a \in S\}$. Then $a \leftrightarrow a - e$ represents a $\ast$-isomorphism between $S$ and $S_1$, $0 = (0, \ldots, 0)$ is the zero of $S_1$. So without loss of generality we can assume that $e = (0, \ldots, 0)$. Let $f = (a_1, \ldots, a_n)$ denote the identity of $S$. So $a_i^2 = a_i$, $i = 1, \ldots, n$. Suppose some $a_i = 0$, say $i = 1$. Then $n > 1$ and $S \subseteq (0) \times K^{n-1}$. So $S$ is $\ast$-isomorphic to a closed subsemigroup of $(K^{n-1}, \cdot)$, contradicting the minimality of $n$. So $a_i \neq 0$, $i = 1, \ldots, n$. So $a_i = 1$, $i = 1, \ldots, n$ and $S$ is a closed submonoid of $(K^n, \cdot)$. This proves (1).

Let $S$ be a closed submonoid of $(K^n, \cdot)$ with identity $1 = (1, \ldots, 1)$, zero $0 = (0, \ldots, 0)$. Let $\chi_i$ denote the $i$th projection of $S$ into $K$. Then clearly $\chi_1, \ldots, \chi_n \in \Phi(S)$. Let $\chi \in \Phi(S)$. Since $\chi(0) = 0$, $\chi$ does not have a constant term. So there exist $a_1, \ldots, a_n \in K$ such that

$$\chi(a) = \sum_{i=1}^{t} a_i \omega_i(a) \text{ for all } a \in S.$$ 

So $x = \sum_{i=1}^{t} a_i \omega_i(x_1, \ldots, x_n)$. By Lemma 2.1(4), $x = \omega_i(x_1, \ldots, x_n)$ for some $i$. So $\Phi(S) = \langle x_1, \ldots, x_n \rangle$. This proves (2). Let $a, b \in S$ such that $\chi(a) = \chi(b)$ for all $\chi \in \Phi(S)$. Then $a = (\chi_1(a), \ldots, \chi_n(a)) = (\chi_1(b), \ldots, \chi_n(b)) = b$. This proves (3).

**Lemma 2.3.** Let $S$ be a closed connected submonoid of $(K^n, \cdot)$ with zero $0 = (0, \ldots, 0)$. Then there exist $u_1, \ldots, u_t, v_1, \ldots, v_t \in F(X_1, \ldots, X_n)$ such that for
a \in K^n, a \in S if and only if \( u_i(a) = v_i(a) \), \( i = 1, \ldots, t \).

PROOF. Let \( x_i \) denote the \( i \)th projection of \( S \) into \( K \). Then \( x_1, \ldots, x_n \in \phi(S) \).

Let \( I = \{ f \mid f \in K[x_1, \ldots, x_n], f(S) = 0 \} \). Let \( D = \{ f \mid f \in I, f = u - v \text{ for some } u, v \in F(x_1, \ldots, x_n) \} \). We claim that \( I = (D) \). Suppose not. Then there exists \( f \in I \), \( f \not\in (D) \). Since \( f(0) = 0 \), there exist \( \omega_1, \ldots, \omega_r \in F(x_1, \ldots, x_n), a_1, \ldots, a_r \in K \setminus \{0\} \) such that \( f = \sum_{i=1}^r a_i \omega_i \). Of all such \( f \) choose one with \( r \) minimal. So \( \sum_{i=1}^r a_i \omega_i(a) = f(a) = 0 \) for all \( a \in S \). So

\[
\sum_{i=1}^r a_i \omega_i(x_1, \ldots, x_n) = 0
\]

By Lemma 2.1(4), \( \omega_p(x_1, \ldots, x_n) = \omega_q(x_1, \ldots, x_n) \) for some \( p, q \in \{1, \ldots, r\}, p \neq q \). Assume \( p = 1, q = 2 \). So \( \omega_1(a) = \omega_2(a) \) for all \( a \in S \). Thus \( \omega_1 - \omega_2 \in D \). Now

\[
(a_1 + a_2)\omega_2 + \sum_{i=3}^r a_i \omega_i = f - a_1(\omega_1 - \omega_2) \in I
\]

By minimality of \( r \), \( f - a_1(\omega_1 - \omega_2) \in (D) \). So \( f \in (D) \), a contradiction. Thus

\[ I = (D) \]. By the Hilbert Basis Theorem there exist \( f_1, \ldots, f_t \in I \) such that

\[ I = (f_1, \ldots, f_t) \]. Since \( I = (D) \), \( f_1, \ldots, f_t \in (D_1) \) for some finite subset \( D_1 \) of \( D \).

So \( (D_1) = I \). This proves the lemma.

Let \( S_1, S_2 \) be connected \( d \)-semigroups with zeros, \( \dim S_1 > 0 \), \( i = 1, 2 \). Let \( \phi: S_1 + S_2 \) be a \( * \)-homomorphism such that \( \phi(1) = 1 \), \( \phi(0) = 0 \). Then define \( \phi^*: \phi(S_2) \to \phi(S_1) \) by \( \phi^*(\chi) = \chi \circ \phi \). Next assume that \( \psi: \phi(S_2) \to \phi(S_1) \) is a homomorphism.

Then we claim:

for all \( a \in S_1 \), there exists unique \( b \in S_2 \) such that

\[
\chi(b) = (\psi(\chi))(a) \text{ for all } \chi \in \phi(S_2)
\]

Assume (6). Then we can define \( \psi: S_1 \to S_2 \) as \( \psi(a) = b \). Then

\[
\chi(\psi(a)) = (\psi(\chi))(a) \text{ for all } a \in S_1, \chi \in \phi(S_2)
\]

Next we claim,

\( \psi \) is a \( * \)-homomorphism, \( \psi(1) = 1, \psi(0) = 0 \)

We now prove (6), (8). Note that the uniqueness of \( b \) in (6) follows from Lemma 2.2(3). By Lemma 2.2(1) we can assume that \( S_2 \) is a closed submonoid of \((K^n, \cdot)\)
with zero $0 = (0, \ldots, 0)$. Let $\chi_i$ denote the $i$th projection of $S_2$ into $K$. Then by Lemma 2.2, $x_1, \ldots, x_n \in \Phi(S_2)$ and $\Phi(S_2) = \langle x_1, \ldots, x_n \rangle$. By Lemma 2.3, there exist $u_1, \ldots, u_t, v_1, \ldots, v_t \in F(x_1, \ldots, x_n)$ such that for $b \in K^n$, $b \in S_2$ if and only if $u_i(b) = v_i(b), i = 1, \ldots, t$. So $u_1(x_1, \ldots, x_n) = v_1(x_1, \ldots, x_n), i = 1, \ldots, t$. Hence $u_i(\psi(x_1), \ldots, \psi(x_n)) = v_i(\psi(x_1), \ldots, \psi(x_n)), i = 1, \ldots, t$. Thus $u_i((\psi(x_1))(a), \ldots, (\psi(x_n))(a)) = v_i((\psi(x_1))(a), \ldots, (\psi(x_n))(a))$ for all $a \in S_1$, $i = 1, \ldots, t$. So $((\psi(x_1))(a), \ldots, (\psi(x_n))(a)) \in S_2$ for all $a \in S_1$. Define $\bar{\psi}: S_1 + S_2$ as $\bar{\psi}(a) = ((\psi(x_1))(a), \ldots, (\psi(x_n))(a))$. So $\bar{\psi}$ is a $*$-homomorphism, $\bar{\psi}(1) = 1$, $\bar{\psi}(0) = 0$. Clearly $\chi_i(\bar{\psi}(a)) = ((\psi(x_i))(a))$ for all $a \in S_i, i = 1, \ldots, t$. Since $\Phi(S_2) = \langle x_1, \ldots, x_n \rangle$, (7) and hence (6) is true. It is clear from (7) that

$$\bar{\psi} = \psi$$

(9)

Now let $\phi: S_1 \to S_2$ be a $*$-homomorphism, $\phi(1) = 1, \phi(0) = 0$. Then for all $\chi \in \Phi(S_2)$, $a \in S_1$, by (7),

$$\chi((\phi(a))) = (\phi(\chi))(a) = \chi(\phi(a))$$

By Lemma 2.2(3),

$$\bar{\phi} = \phi$$

(10)

THEOREM 2.4. Let $S_1, S_2, S_3$ be connected $d$-semigroups with zeros, dim $S_i > 0$, $i = 1, 2, 3$. Then

1. If $\phi: S_1 \to S_2$ is a $*$-homomorphism with $\phi(0) = 0, \phi(1) = 1$, then

$$\phi^* : \Phi(S_2) \to \Phi(S_1)$$

is a homomorphism and $\phi^* = \phi$.

2. If $\iota: S_1 \to S_1$ is the identity map then $\iota^* : \Phi(S_1) \to \Phi(S_1)$ is the identity map.

3. If $\psi: \Phi(S_2) \to \Phi(S_1)$ is a homomorphism, then $\bar{\psi}: S_1 + S_2$ is a $*$-homomorphism with $\bar{\psi}(0) = 0, \bar{\psi}(1) = 1$. Moreover $\bar{\psi}^* = \psi$.

4. If $\iota: \Phi(S_1) \to \Phi(S_1)$ is the identity map, then $\iota: S_1 + S_1$ is the identity map.

5. If $\phi_1: S_1 \to S_2, \phi_2: S_2 \to S_3$ are $*$-homomorphism with $\phi_i(0) = 0, \phi_i(1) = 1$, $i = 1, 2$, then $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*$. 


If \( \psi_1: \phi(S_2) \rightarrow \phi(S_1), \psi_2: \phi(S_3) \rightarrow \phi(S_2) \) are homomorphisms, then \( \psi_1 \circ \psi_2 = \psi_2 \circ \psi_1 \).

(7) \( S_1 \) is *-isomorphic to \( S_2 \) if and only if \( \phi(S_1) \) is isomorphic to \( \phi(S_2) \).

**Proof.** (1), (3) follow from the equations (6)-(10). (2), (4) are trivial. (7) follows from (2), (4), (5), (6). So we need only prove (5), (6). First we prove (5). Let \( \chi \in \phi(S_3) \). Then for all \( a \in S_1 \),

\[
((\phi_2 \circ \phi_1) \ast)(\chi)(a) = \chi(\phi_2(\phi_1(a)))
\]

\[
= (\phi_2(\chi))(\phi_1(a))
\]

\[
= ((\phi_1 \ast \phi_2)(\chi))(a)
\]

Next we prove (6). Let \( a \in S_1 \), \( \chi \in \phi(S_3) \). Then by equation (7),

\[
\chi(\psi_1 \circ \psi_2(a)) = ((\psi_1 \circ \psi_2)(\chi))(a)
\]

\[
= (\psi_1(\psi_2(\chi)))(a)
\]

\[
= (\psi_2(\chi))(\psi_1(a))
\]

\[
= \chi(\psi_2(\psi_1(a)))
\]

\[
= \chi(\psi_1(a))
\]

By Lemma 2.2(3), \( \psi_1 \circ \psi_2 = \psi_2 \circ \psi_1 \), proving the theorem.

**Theorem 2.5.** Let \( e \in K \) \( F(X_1, \ldots, X_m) \). Let \( V = \{ (e_1(a), \ldots, e_n(a)) | a \in K^n \} \)

\( \subseteq K^n \). Set \( S = \bar{V} \). Then \( S \) is a closed connected \( d \)-submonoid with zero, of \( (K^n, \ast) \).

Moreover \( \phi(S) \) \( \ast < \omega_1, \ldots, \omega_n > \).

**Proof.** Define \( \theta: (K^n, \ast) \rightarrow (K^n, \ast) \) as \( \theta(a) = (\omega_1(a), \ldots, \omega_n(a)) \). Then \( \theta \) is a *-homomorphism with image \( V \). So \( S = \bar{V} \) is connected. Clearly \( 1 = \theta(1), 0 = \theta(0) \in S \).

Let \( \chi_i \) denote the \( i \)-th projection of \( S \) into \( K \). Then by Lemma 2.2, \( \phi(S) = < x_1, \ldots, x_n > \).

Let \( u, v \in F(Y_1, \ldots, Y_n) \). Suppose \( u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \).

Then \( u(b) = v(b) \) for all \( b \in S \). So
\[ u(\omega_1(a), \ldots, \omega_n(a)) = v(\omega_1(a), \ldots, \omega_n(a)) \quad \text{for all } a \in K^m \]  

Since \( K \) is infinite, \( u(\omega_1, \ldots, \omega_n) = v(\omega_1, \ldots, \omega_n) \) in \( F(X_1, \ldots, X_m) \). Conversely suppose \( u(\omega_1, \ldots, \omega_n) = v(\omega_1, \ldots, \omega_n) \) in \( F(X_1, \ldots, X_m) \). Then (11) is true. So \( u(b) = v(b) \) for all \( b \in V \). Since \( \overline{V} = S \), \( u(b) = v(b) \) for all \( b \in S \). So \( u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \). It follows that \( \phi(S) = \langle x_1, \ldots, x_n \rangle \preceq \langle \omega_1, \ldots, \omega_n \rangle \).

By Theorem A, Lemmas 2.1, 2.2, Theorems 2.4, 2.5, we have.

**Theorem 2.6.** Let \( N_1 \) be the category of connected \( d \)-semigroups with zeros of dimension \( > 0 \) with morphism being \( * \)-homomorphisms with \( \phi(0) = 0, \phi(1) = 1 \). Let \( N_2 \) be the category of finitely generated, commutative, idempotent free, totally cancellative semigroups with morphisms being semigroup homomorphisms. Then \( (\phi, *) \) is a contravariant equivalence between \( N_1 \) and \( N_2 \).

**Theorem 2.7.** Let \( S \) be a closed connected submonoid of \( (K^n, \cdot) \) with zero \( 0 = (0, \ldots, 0) \). Then for some \( m \in \mathbb{Z}^+ \), \( \omega_1, \ldots, \omega_n \in F(X_1, \ldots, X_m) \), \( S = \overline{V} \) where \( V = \{(\omega_1(a), \ldots, \omega_n(a)) | a \in K^m\} \).

**Proof.** Let \( x_i \) denote the \( i \)-th projection of \( S \) into \( K \). Then by Lemma 2.2, \( \phi(S) = \langle x_1, \ldots, x_n \rangle \). By Theorem A, \( \phi(S) \preceq \langle \omega_1, \ldots, \omega_n \rangle \) for some \( m \in \mathbb{Z}^+ \), \( \omega_1, \ldots, \omega_n \in F(X_1, \ldots, X_m) \) with \( \chi_1 \leftrightarrow \omega_1 \). Let \( V = \{(\omega_1(a), \ldots, \omega_n(a)) | a \in K^m\} \) and set \( S_1 = \overline{V} \). Then \( 1 = (1, \ldots, 1), 0 = (0, \ldots, 0) \in S_1 \). Let \( u, v \in F(Y_1, \ldots, Y_n) \). Suppose \( u(c) = v(c) \) for all \( c \in S \). Then \( u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \). So \( u(\omega_1, \ldots, \omega_n) = v(\omega_1, \ldots, \omega_n) \). Thus \( u(b) = v(b) \) for all \( b \in V \). Since \( \overline{V} = S_1 \), \( u(b) = v(b) \) for all \( b \in S_1 \). Conversely suppose \( u(b) = v(b) \) for all \( b \in S_1 \). Then

\[ u(\omega_1(a), \ldots, \omega_n(a)) = v(\omega_1(a), \ldots, \omega_n(a)) \quad \text{for all } a \in K^m \]

Since \( K \) is infinite, \( u(\omega_1, \ldots, \omega_n) = v(\omega_1, \ldots, \omega_n) \). So \( u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \).

Thus \( u(c) = v(c) \) for all \( c \in S \). By Lemma 2.3, \( S = S_1 \).

**Corollary 2.8.** Let \( S \) be a closed connected submonoid of \( (K^n, \cdot) \) with zero \( 0 = (0, \ldots, 0) \), \( \dim S = 1 \). Then there exist \( i_1, \ldots, i_n \in \mathbb{Z}^+ \) such that \( S = \langle a_1, \ldots, a_n \rangle | a \in K \rangle \). Conversely, for any \( i_1, \ldots, i_n \in \mathbb{Z}^+ \), \( S \) defined as above is a closed connected submonoid of \( (K^n, \cdot) \) with zero \( 0 = (0, \ldots, 0) \) and \( \dim S = 1 \).
V = \{ (ω_1(a),...,ω_n(a)) | a \in K^n \} and S = \overline{V}. Let V_1 = \{ (ω_1(a,...,a),...,ω_n(a,...,a)) | a \in K \}, S_1 = \overline{V_1}. Then S_1 \subseteq S, \dim S_1 = 1. So S = S_1. So there exist i_1,...,i_n \in \mathbb{Z}^+ such that V_1 = \{ (a^1_1,...,a^n_1) | a \in K \}. Define \( \theta : K \to S \) as \( \theta(a) = (a^1,...,a^n) \). Then it is easy to see that \( \theta \) is a finite morphism in the usual sense of [5; Section 4.2]. By [5; Proposition 4.2], \( \theta(K) = S \).

**THEOREM 2.9.** Let \( S \) be a connected monoid with zero, \( \dim S = 1 \). Then \( S \) is *-isomorphic to a semigroup of the type given in Corollary 2.8.

**PROOF.** By Corollary 1.5, \( S \) is a d-semigroup. We are now done by Lemma 2.2 and Corollary 2.8.

**THEOREM 2.10.** Let \( S \) be a connected semigroup, \( e, f \in E(S) \), \( e \geq f \). Then there exists a closed connected subsemigroup \( S_1 \) of \( S \) such that \( e \) is the identity of \( S_1 \), \( f \) is the zero of \( S_1 \) and \( \dim S_1 = 1 \).

**PROOF.** We can assume that \( e \) is the identity element of \( S \) (otherwise we work with \( eSe \)). By Lemma 1.1 we are reduced to the case when \( f \) is the zero of \( S \). By Corollary 1.5, we are reduced to the case when \( S \) is also a d-semigroup.

\[
\begin{align*}
A(S) &= \{ \text{All prime ideals of } S \} \cup \{ \emptyset \}. \\
X(S) &= \{ \text{All } I \subseteq I(S) \} \\
\Omega(S) &= \text{Maximal semilattice image of } S.
\end{align*}
\]

It is easy to see that \( (A(S), \subseteq) \cong (A(\Omega(S)), \subseteq) \) is a complete lattice. If \( S \) is finitely generated, then \( \Omega(S) \) is finite and so \( (A(S), \subseteq) \) is a finite lattice.

**THEOREM 3.1.** Let \( S \) be a connected d-semigroup with zero. Define \( \alpha : I(S) \to \Gamma(\emptyset(S)) \) as \( \alpha(I) = \{ \chi | \chi \in \emptyset(S), \chi(a) = 0 \text{ for all } a \in I \} \). Define \( \beta : \Gamma(\emptyset(S)) \to I(S) \) as \( \beta(W) = \{ a | a \in S, \chi(a) = 0 \text{ for all } \chi \in W \} \). Then \( \alpha, \beta \) are inclusion reversing bijections and \( \beta = \alpha^{-1} \). Moreover \( \alpha(\Omega(S)) = \Omega(\emptyset(S)) \).

**PROOF.** Clearly \( \alpha, \beta \) are inclusion reversing. Let \( I \in A(S) \). Then \( I = eS \) for some \( e \in E(S) \). So \( \alpha(I) = \{ \chi | \chi \in \emptyset(S), \chi(e) = 0 \} \). It follows that \( \alpha(I) \in A(\emptyset(S)) \).

Clearly \( I \subseteq \beta(\alpha(I)) \). We claim that \( I = \beta(\alpha(I)) \). Suppose not. Then there exists \( a \in \beta(\alpha(I)) \) such that \( a \notin I \). Let \( h f, f \in E(S) \). Then \( f \notin I, f \in \beta(\alpha(I)) \). So \( e \neq f \). By Lemma 2.1(2), there exists \( \chi \in \emptyset(S) \) such that \( \chi(f) = 1, \chi(e) = 0 \). So
\( \chi \in a(I) \) and \( f \not\in B(a(I)) \), a contradiction. So

for all \( I \in A(S) \), \( a(I) \in A(\phi(S)) \) and \( B(a(I)) = I \) \hfill (12)

Let \( P \in A(\phi(S)) \). We claim that \( B(P) \in A(S) \) and \( a(B(P)) = P \). By Lemma 2.1, this is true for \( P = \phi(S) \). So assume \( P \neq \phi(S) \). Then \( F = \phi(S) \setminus P \) is a subsemigroup of \( \phi(S) \). By Lemma 2.2 we can assume that \( S \) is a closed submonoid of some \((K^n, \cdot)\), \( 0 = (0, \ldots, 0) \in S \) and that \( \phi(S) = \langle x_1, \ldots, x_n \rangle \) where \( x_i \) is the \( i \)th projection of \( S \) into \( K_i \), \( i = 1, \ldots, n \). Let \( A = \{x_i | x_i \in F \} \). Then \( \langle A \rangle = F \). Let \( e = (e_1, \ldots, e_n) \) where \( e_1 = 1 \) if \( x_i \in A \), \( e_0 = 0 \) if \( x_i \not\in A \). We claim that \( e \in S \). Suppose not. Then by Lemma 2.3, there exist \( u, v \in F(x_1, \ldots, x_n) \) such that \( u(a) = v(a) \) for all \( a \in S \) and \( u(e) \neq v(e) \). Since \( u(e)^2 = u(e) \) and \( v(e)^2 = v(e) \) we can assume that \( u(e) = 1 \), \( v(e) = 0 \). Clearly \( u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \). Since \( u(e) = 1 \), \( u(x_1, \ldots, x_n) \)

By Lemma 2.2 and Theorem 2.7, we can assume that \( S \) is as in Theorem 2.7, with \( e = (1, \ldots, 1) \), \( f = (0, \ldots, 0) \). Let \( V_1 = \{(\omega_1(a, \ldots, a), \ldots, \omega_n(a, \ldots, a)) | a \in K\} \), \( S_1 = V_1 \). Then \( e, f \in S_1 \), \( \dim S_1 = 1 \), \( S_1 \subseteq S \). Define \( \theta : K \to S_1 \) as \( \theta(a) = (\omega_1(a, \ldots, a), \ldots, \omega_n(a, \ldots, a)) \). Then \( \theta \) is a \( * \)-homomorphism. So \( S_1 \) is connected. This proves the theorem.

3. POLYTOPES

If \( X \subseteq \mathbb{R}^n \), then we let \( C(X) \) denote the convex hull of \( X \) (see [4]). The convex hull of a finite set in \( \mathbb{R}^n \) is called a polytope [4]. If the vertices of \( P \) are rational, then \( P \) is said to be a rational polytope. If \( X \subseteq P \), then \( X \) is said to be a face of \( P \) [4; p. 35] if for all \( a, b \in P \), \( a \in (0,1) \), \( a \alpha + (1-\alpha) b \in X \) if and only if \( a, b \in X \). Let \( X(P) \) denote the set of all faces of \( P \). Then [4; p. 21], \( (X(P), \subseteq) \) is a finite lattice. Dimension of \( P \) is defined to be the dimension of the affine hull of \( P \) [4; p. 3]. Then dimension of \( P \) = (length of any maximal chain in \( X(P) \)) - 1. Two polytopes \( P_1, P_2 \) have the same combinatorial type if \( X(P_1) \sim X(P_2) \) (see [4; p. 38]). By [4; p. 244], every polytope of dimension \( \leq 3 \) has the same combinatorial type as some rational polytope. However this is not true in general [4; p. 94]. If \( u = (a_1, \ldots, a_n) \), \( v = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \) then let \( u \cdot v = \sum_{i=1}^{n} a_i \beta_i \) denote the inner product of \( u \) and \( v \).
Let $S$ be a semigroup. An ideal $I$ of $S$ is said to be \textit{semiprime} if for all $a \in S$, $a^2 \in I$ implies $a \in I$. $I$ is \textit{prime} if for all $a$, $b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. Let

$$ I(S) = \{ \text{All ideals of } S \} $$

$$ A(S) = \{ \text{All principal ideals of } S \} $$

$$ I(S) = \{ \text{All semiprime ideals of } S \} \cup \{ \emptyset \} $$

involves only those $X_i$'s with $X_i \in F$. So $u(x_1, \ldots, x_n) \in F$. Since $v(e) = 0$, $v(x_1, \ldots, x_n)$ involves at least one $X_i$ with $X_i \notin F$. So $v(x_1, \ldots, x_n) \in P$. This contradiction shows that $e \in S$. Clearly $\chi(e) = 1$ for $\chi \in F$, $\chi(e) = 0$ for $\chi \in P$. Hence $P = \{ \chi | \chi \in \phi(S), \chi(e) = 0 \} = \alpha(eS)$. By (12), $\beta(P) = \beta(\alpha(eS)) = eS \in A(S)$, $\alpha(\beta(P)) = \alpha(eS) = P$. So

for all $P \in A(\phi(S))$, $\beta(P) \in A(S)$ and $\alpha(\beta(P)) = P$ \hspace{1cm} (13)

Clearly

$$ \alpha(I_1 \cup I_2) = \alpha(I_1) \cap \alpha(I_2) \text{ for all } I_1, I_2 \in I(S) \hspace{1cm} (14) $$

Let $W_1, W_2 \in \Gamma(\phi(S))$. Then clearly $\beta(W_1) \subseteq \beta(W_1 \cap W_2)$, $i = 1, 2$. So $\beta(W_1) \cup \beta(W_2) \subseteq \beta(W_1 \cap W_2)$. Let $a \in \beta(W_1 \cap W_2)$. Suppose $a \notin \beta(W_i)$, $i = 1, 2$. Then there exist $\theta_i \in W_i$, $i = 1, 2$ such that $\theta_i(a) \neq 0$, $i = 1, 2$. So

$$ \theta = \theta_1 \theta_2 \in W_1 \cap W_2, \theta(a) \neq 0. \text{ So } a \notin \beta(W_1 \cap W_2), \text{ a contradiction. Thus} $$

$$ \beta(W_1 \cap W_2) = \beta(W_1) \cup \beta(W_2) \text{ for all } W_1, W_2 \in \Gamma(\phi(S)) \hspace{1cm} (15) $$

Clearly $A(S)$ is finite. Let $I \in I(S)$. Then $I = I_1 \cup I_2 \cup \ldots \cup I_k$ for some $I_1, \ldots, I_k \in A(S)$. By (12), $\beta(\alpha(I)) = I_r$, $r = 1, \ldots, k$. By (14), (15),

$$ \alpha(I) = \alpha(I_1) \cap \ldots \cap \alpha(I_k) $$

$$ \beta(\alpha(I)) = \beta(\alpha(I_1)) \cup \ldots \cup \beta(\alpha(I_k)) $$

$$ = I_1 \cup \ldots \cup I_k $$

$$ = I $$

So

$$ \beta(\alpha(I)) = I \text{ for all } I \in I(S) \hspace{1cm} (16) $$

Since $\phi(S)$ is finitely generated, $A(\phi(S))$ is finite. Let $W \in \Gamma(\phi(S))$. By [2; p. 125, Exercise 9], $W = W_1 \cap \ldots \cap W_k$ for some $W_1, \ldots, W_k \in A(\phi(S))$. Then by (13)
\[ \alpha(\beta(W)) = W \text{ for } r = 1, \ldots, k. \] Then by (14), (15),

\[
\begin{align*}
\beta(W) &= \beta(W_1) \cup \ldots \cup \beta(W_k) \\
\alpha(\beta(W)) &= \alpha(\beta(W_1)) \cap \ldots \cap \alpha(\beta(W_k)) \\
&= W_1 \cap \ldots \cap W_k \\
&= W
\end{align*}
\]

So

\[ \alpha(\beta(W)) = W \text{ for all } W \in \Gamma(\Phi(S)) \quad (17) \]

By (16) and (17), \( \alpha^{-1} = \beta \). By (12), (13), \( \alpha(A(S)) = A(\Phi(S)) \). This proves the theorem.

**REMARK.** The classical Hilbert's Nullstellensatz yields a 1-1 correspondence between the closed subsets of \( \mathbb{K}^n \) and the radical ideals of \( \mathbb{K}[X_1, \ldots, X_n] \). Moreover this restricts to a 1-1 correspondence between the closed irreducible subsets of \( \mathbb{K}^n \) and the prime ideals of \( \mathbb{K}[X_1, \ldots, X_n] \). Analogously, Theorem 3.1 yields a 1-1 correspondence between the ideals of a connected d-semigroup with zero \( S \) and the semiprime ideals of its character semigroup \( \Phi(S) \). Moreover this correspondence restricts to a correspondence between the principal ideals of \( S \) and the prime ideals of \( \Phi(S) \).

**THEOREM 3.2.** Let \( S \) be a connected d-semigroup with zero. Then

\[ (U(S), \leq) \preceq (E(S), \leq) \preceq (X(\Phi(S)), \subseteq). \]

**PROOF.** Clearly \( (A(S), \subseteq) \preceq (U(S), \leq) \preceq (E(S), \leq) \). By Theorem 3.1, \( (A(S), \subseteq) \) is anti-isomorphic to \( (A(\Phi(S)), \subseteq) \). Clearly \( (A(\Phi(S)), \subseteq) \) is anti-isomorphic to \( (X(\Phi(S)), \subseteq) \). This proves the theorem.

Let \( \mathbb{Q}^{m \times n} \) denote the set of all \( m \times n \) matrices over \( \mathbb{Q} \). The following result is well known. However, we include a proof here for the convenience of the reader.

**FACT 3.3.** Let \( A \in \mathbb{Q}^{m \times n}, u = (a_1, \ldots, a_m) \in \mathbb{R}^m \) such that \( uA = 0 \). Then there exists \( v = (\beta_1, \ldots, \beta_m) \in \mathbb{Q}^m \) such that \( vA = 0 \) and for \( i = 1, \ldots, m, a_i > 0 \) implies \( \beta_i > 0 \), \( a_i < 0 \) implies \( \beta_i < 0 \).

**PROOF.** Let \( N = \{ X | X \in \mathbb{R}^m, XA = 0 \} \) denote the left null space of \( A \). Since \( A \in \mathbb{Q}^{m \times n} \), there exist \( u_1, \ldots, u_t \in \mathbb{Q}^m \) such that \( u_1, \ldots, u_t \) is a basis of \( N \). So
Let \( u = \sum_{j=1}^{t} \epsilon_j u_j \) for some \( \epsilon_1, \ldots, \epsilon_t \in \mathbb{R} \). Let \( \epsilon \in \mathbb{R}^+ \), \( \epsilon_1', \ldots, \epsilon_t' \in \mathbb{Q} \). Then
\[
v = \sum_{j=1}^{t} \epsilon_j' u_j \in \mathbb{N} \cap \mathbb{Q}^m.
\]
For \( \frac{\epsilon_j'}{\epsilon_j} - 1 \) small enough, \( |u-v| < \epsilon \). For \( \epsilon \) small enough, the conclusion of the lemma clearly holds.

**COROLLARY 3.4.** Let \( u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{Q}^d \), \( \alpha_1, \ldots, \alpha_m \), \( \beta_1, \ldots, \beta_n \in \mathbb{R}^+ \) such that \( \sum_{i=1}^{m} \alpha_i u_i = \sum_{j=1}^{n} \beta_j v_j \). Then there exist \( \alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n \in \mathbb{Z}^+ \) such that
\[
\sum_{i=1}^{m} \alpha'_i u_i = \sum_{j=1}^{n} \beta'_j v_j.
\]

**PROOF.** By Fact 3.3 we can choose \( \alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n \in \mathbb{Q}^+ \) such that
\[
\sum_{i=1}^{m} \alpha'_i u_i = \sum_{j=1}^{n} \beta'_j v_j.
\]
Then for some \( s \in \mathbb{Z}^+ \), \( \alpha'_i = s \alpha_i \), \( \beta'_j = s \beta_j \in \mathbb{Z}^+ \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \). Clearly \( \sum_{i=1}^{m} \alpha'_i u_i = \sum_{j=1}^{n} \beta'_j v_j \).

**THEOREM 3.5.** The classes \( \{X(S) \mid S \text{ is a finitely generated, commutative, idempotent-free, totally cancellative semigroup}\} \) and \( \{X(P) \mid P \text{ is a rational polytope in } \mathbb{R}^n \text{ for some } n \in \mathbb{Z}^+\} \) are identical to within lattice isomorphisms.

**PROOF.** Let \( S \) be a finitely generated, commutative, idempotent-free, totally cancellative semigroup. By Theorem A we can assume that \( S = \langle u_1, \ldots, u_n \rangle \subseteq (\mathbb{Z}^d)^+, \) \( 0 \notin S \). Let \( C = C(u_1, \ldots, u_n) \). By Fact 3.3, \( 0 \notin C \). So \( C \cap -C = \emptyset \). By [4; p. 11], there exists \( u \in \mathbb{R}^d \) such that \( u \cdot a > 0 \) for all \( a \in C \). So \( u \cdot u_i > 0 \), \( i = 1, \ldots, n \). If \( v \in \mathbb{R}^d \), then \( |u \cdot u_i - v \cdot u_i| = |(u - v) \cdot u_i| < ||u - v|| \cdot ||u_i|| \). So for \( ||u - v|| \) small enough, \( v \cdot u_i > 0 \) for \( i = 1, \ldots, n \). So, without loss of generality, we can assume that \( u \in \mathbb{Q}^d \). If \( a \in S \), then let \( \theta(a) = \frac{a}{u \cdot a} \in \mathbb{Q}^d \). Let \( a_1, \ldots, a_k \in S \), \( a_1 \ldots a_k \in \mathbb{Z}^+ \) and set \( a = a_1 a_1 + \ldots + a_k a_k \). Then
\[
\theta(a) = \sum_{i=1}^{k} \beta_i \theta(a_i) \in C(\theta(a_1), \ldots, \theta(a_k)),
\]
where \( \beta_i = \frac{a_i a_i \cdot u}{a \cdot u} > 0 \), \( i = 1, \ldots, k \).
So $P = C(\theta(S)) = C(\theta(u_1), \ldots, \theta(u_n))$ is a rational polytope. If $X \in X(S)$, then let $\phi(X) = C(\theta(X)) \subseteq P$. If $F \in X(P)$, then let $\psi(F) = \{a|a \in S, \theta(a) \in F\} \subseteq S$.

Let $X \in X(S)$. Let $x, y \in P$, $a \in (0,1)$ such that $ax + (1-a)y = z \in \phi(X)$. There exist $a_1, \ldots, a_p, b_1, \ldots, b_q \in S$, $c_1, \ldots, c_r \in X$, $a_1, \ldots, a_p, \beta_1, \ldots, \beta_q$, $\gamma_1, \ldots, \gamma_r \in (0,1)$ such that $x = \sum a_i \theta(a_i)$, $y = \sum b_j \theta(b_j)$, $z = \sum \gamma_k \theta(c_k)$, $\sum a_i = \sum b_j = 1$. So there exist $a_1', \ldots, a_p', \beta_1', \ldots, \beta_q', \gamma_1', \ldots, \gamma_r' \in \mathbb{R}^+$ such that

$$\sum a_i'a_i + \sum b_j'b_j = \sum \gamma_k'c_k \in X$$

By Corollary 3.4 there exist $a_1'', \ldots, a_p'', \beta_1'', \ldots, \beta_q'', \gamma_1'', \ldots, \gamma_r'' \in \mathbb{Z}^+$ such that

$$\sum a_i''a_i + \sum b_j''b_j = \sum \gamma_k''c_k \in X$$

Since $X \in X(S)$, $a_1, \ldots, a_p, b_1, \ldots, b_q \in X$. Since $x \in C(\theta(a_1), \ldots, \theta(a_p))$ and $y \in C(\theta(b_1), \ldots, \theta(b_q))$, $x, y \in \phi(X)$. Hence $\phi(X) \in X(F)$. Clearly $X \subseteq \psi(\phi(X))$.

Let $a \in \psi(\phi(X))$. Then $\theta(a) \in C(\theta(X))$. So there exist $a_1, \ldots, a_p \in X$, $\theta(a_1), \ldots, \theta(a_p)$ such that $\theta(a) = \sum a_i \theta(a_i)$. By Corollary 3.4, there exist $a_1', \ldots, a_p' \in \mathbb{Z}^+$ such that $\theta(a) = \sum a_i'a_i \in X$. So $a \in X$. Hence

for all $X \in X(S)$, $\phi(X) \in X(P)$ and $\psi(\phi(X)) = X$ \hspace{1cm} (19)

Let $F \in X(P)$. By (18) $\psi(F)$ is $\emptyset$ or a subsemigroup of $S$. Let $a, b \in S$ such that $a + b \in \psi(F)$. By (18), $\theta(a+b) = \varepsilon \theta(a) + (1-\varepsilon) \theta(b) \in F$ for some $\varepsilon \in (0,1)$. So $\theta(a), \theta(b) \in F$. Hence $a, b \in \psi(F)$ and $\psi(F) \in X(S)$. Clearly $\psi(\psi(F)) \subseteq F$. Let $x \in F$. Then $x \in \phi(S)$. So $x = \sum_{i=1}^{k} \varepsilon_i \theta(a_i)$ for some $a_1, \ldots, a_k \in S$, $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{Q}$ such that $\sum \varepsilon_i = 1$. Then $\theta(a_i) \in F$, $i = 1, \ldots, k$. So $a_1, \ldots, a_k \in \psi(F)$ and $x \in \phi(\psi(F))$. So $\phi(\psi(F)) = F$. Hence

for all $F \in X(P)$, $\psi(F) \in X(S)$ and $\phi(\psi(F)) = F$ \hspace{1cm} (20)

Since $\phi, \psi$ are clearly inclusion preserving, it follows from (19), (20) that

$(X(S), \subseteq) \preceq (X(P), \subseteq)$.

Conversely let $P \subseteq \mathbb{R}^m$ be a rational polytope. Then $P = C(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in \mathbb{Q}^m$. If $a \in \mathbb{Z}^+$, then clearly $(X(P), \subseteq) \preceq (X(aP), \subseteq)$. So we can assume that $a_1, \ldots, a_n \in \mathbb{Z}^m$. Let $u_i = (a_i, 1)$, $i = 1, \ldots, n$, $d = m+1$. Then $P_1 = C(u_1, \ldots, u_n)=$
P × {1} ⊆ \mathbb{R}^d is a rational polytope and \((X(P), \subseteq) \cong (X(P_1), \subseteq)\). Let S = \(< u_1, \ldots, u_n > \subseteq \mathbb{Z}^d\). Then 0 ∉ S and S is a finitely generated, commutative, totally cancellative, idempotent-free semigroup. Let \(u = (0,1) \in \mathbb{Z}^d\). Then \(u \cdot u_i = 1\), \(i = 1, \ldots, n\). So \(\theta(u_i) = \frac{u_i}{u \cdot u_i} = u_i\), \(i = 1, \ldots, n\). By the proof of the first half of this theorem, \((X(S), \subseteq) \cong (X(P_1), \subseteq)\). This proves the theorem.

If S is a connected d-semigroup with zero, then by [7; Theorem 3.17], dim S = length of any maximal chain in E(S). By Theorems 2.6, 3.2 and 3.5 we have,

**THEOREM 3.6.** The classes \(\{(E(S), \subseteq) \mid S \text{ is a connected d-semigroup with zero, dim } S > 0\}\) and \(\{(X(P), \subseteq) \mid P \text{ is a rational polytope in } \mathbb{R}^n \text{ for some } n \in \mathbb{Z}^+\}\) are identical to within lattice isomorphisms. Moreover, for any corresponding S and

**EXAMPLE 3.7.** If \(S = (K^4, \cdot)\), then the corresponding polytope P is a tetrahedron

More generally if \(S = (K^n, \cdot)\), then the corresponding polytope P is the \((n-1)\)-simplex.

**EXAMPLE 3.8.** Let \(S = \{(a_1, b_1, a_2, b_2, a_3, b_3) \mid a_i, b_j \in K, a_i b_j = a_j b_i, i, j = 1, 2, 3\}\). Then by [7; Example 4.7] S is a closed connected d-submonoid with zero, of \((K^6, \cdot)\). Moreover dim S = 4 and \(|E(S)| = 22\). The corresponding polytope P can be shown to be the triangular prism:

**EXAMPLE 3.9.** Let \(S = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_1, \ldots, a_6 \in K, a_3 a_1 = a_2 a_2, a_2 a_5 = a_1 a_4, a_2 a_4 = a_0 a_3\}\). Define \(\phi: k^6 \to k^6\) as

\[\phi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1^2 x_9 x_5, x_2^2 x_3 x_5, x_1^2 x_2 x_5, x_1^2 x_2 x_4, x_1^2 x_4 x_5, x_6)\]

Then \(\phi(k^6) = S\) and so S is a closed connected d-submonoid with zero, of \((K^6, \cdot)\). Clearly dim S = 4 and \(|E(S)| = 24\). The corresponding polytope P can be shown to be the pentagonal pyramid:
COROLLARY 3.10. Let $S$ be a connected semigroup such that $U(S)$ is the following lattice:

Then $n \leq 2$.

**PROOF.** By the proof of Theorem 1.9, we can assume that $S$ is a monoid. Let $S_1$ be a maximal connected $d$-submonoid with zero, of $S$. Then by Theorem 1.8(3) $E(S_1) \cap J \neq \emptyset$ for all $J \in U(S)$ and $\dim S_1 = 2$. By Theorem 3.6, the polytope $P$ corresponding to $S_1$ has dimension 1. So $P$ is the line

So $|E(S_1)| = 4$. Thus $|U(S)| \leq |E(S_1)| = 4$. Hence $n \leq 2$.

**EXAMPLE 3.11.** $M_2(K)$ and $(K^2, \cdot)$ show that $n$ can be 1 or 2 in Corollary 3.10.

4. **SEMILATTICES**

As usual, by a semilattice, we mean a commutative, idempotent semigroup.

**LEMMA 4.1.** Let $\Omega$ be a finite semilattice. Then $|X(\Omega)| = |\Omega| + 1$.

**PROOF.** We prove by induction on $|\Omega|$. If $|\Omega| = 1$ this is clear. So assume $|\Omega| > 1$. Let $\alpha$ be a maximal element of $\Omega$. Then $\{\alpha\} \in X(\Omega)$. Define $\phi: X(\Omega) \to X(\Omega_1)$ as $\phi(F) = F \cap \Omega_1$. Let $F_1 \in X(\Omega_1)$ and set $P_1 = \Omega_1 \setminus F_1$. Let $p \in P_1$. We claim that $\alpha p \in P_1$. Otherwise $f = \alpha p \in F_1$. Then $f = pf \in P_1$, a contradiction. So $\alpha p \subseteq P_1$.

If $F_1 \in X(\Omega)$, then $\phi(F_1) = F_1$. Suppose not. Then $\alpha F_1 \not\subseteq P_1$. So there exists $f_1 \in F_1$ such that $\alpha f_1 \in F_1$. Now we claim that $F_1 \cup \{\alpha\} \in X(\Omega)$. Otherwise $\alpha F_1 \not\subseteq F_1$. So $f_2 \in F_1$ for some $f_2 \in F_1$. So $\alpha f_2 = (\alpha f_1) f_2 \in F_1$ and $\alpha f_2 = (\alpha f_2) f_1 \in P_1$, a contradiction. So $F_1 \cup \{\alpha\} \in X(\Omega)$, $\phi(F_1 \cup \{\alpha\}) = F_1$. Thus $\phi$ is surjective. Let $F_1 \in X(\Omega_1), F_1 \neq \emptyset, F, G \in X(\Omega), \phi(F) = F_1 = \phi(G), F \neq G$. We can assume that $\alpha \in F, \alpha \not\in G$. So $G = F_1, F = F_1 \cup \{\alpha\}$. Since $\alpha \in F$, $\alpha F \subseteq F$. So $\alpha F_1 \subseteq F_1$. 


Since \( a \not\in G \), \( aG \subseteq \Omega \setminus G \). So \( dF_1 \subseteq \Omega_1 \setminus F_1 \), a contradiction. Thus \( |\phi^{-1}(F_1)| = 1 \).

Clearly \( \phi^{-1}(\emptyset) = \{\emptyset, \{a\}\} \). So \( |X(\Omega)| = |X(\Omega_1)| + 1 = |\Omega_1| + 1 + 1 = |\Omega| + 1 \).

If \( \Omega \) is a semilattice, then let \( V(\Omega) \) denote the semilattice of all homomorphisms of \( \Omega \) into \( \Omega_0 = \{0,1\} \). Then clearly \( V(\Omega) \cong (X(\Omega), \cap) \). Let \( V^*(\Omega) = V(\Omega) \setminus \{1,0\} \).

Then \( V^*(\Omega) \) may or may not be a subsemilattice of \( V(\Omega) \).

**Lemma 4.2.** Let \( \Omega \) be a finite semilattice. Then \( V^*(V(\Omega)) \) is a semilattice and \( \Omega \cong V^*(V(\Omega)) \).

**Proof.** Define \( \theta: \Omega + V(V(\Omega)) \) as \( \theta(a)(f) = f(a) \). Then \( \theta \) is a homomorphism. Clearly \( \theta(a)(1) = 1 \), \( \theta(a)(0) = 0 \). So \( \theta(a) \in V^*(V(\Omega)) \). We claim that \( \theta \) is injective. We can assume that \( \Omega \subseteq \Omega_0 \times \ldots \times \Omega_0 \). Let \( f_i \) denote the \( i \)th projection of \( \Omega \) into \( \Omega_0 \). Then \( f_i \in V(\Omega) \). Let \( a, b \in \Omega \) such that \( \theta(a) = \theta(b) \). Then \( \theta(a)(f_i) = \theta(b)(f_i) \) for all \( i \). So \( f_i(a) = f_i(b) \) for all \( i \). So \( a = b \). By Lemma 4.1, \( |V^*(V(\Omega))| = |\Omega| \). Hence \( \Omega \cong V^*(V(\Omega)) \).

**Corollary 4.3.** Let \( \Omega_1, \Omega_2 \) be finite semilattice such that \( (X(\Omega_1), \subseteq) \cong (X(\Omega_2), \subseteq) \). Then \( \Omega_1 \cong \Omega_2 \).

**Corollary 4.4.** Let \( \Omega \) be a finite semilattice such that \( V^*(\Omega) \) is a semilattice. Then \( V(V^*(\Omega)) \cong \Omega \).

**Proof.** Let \( \Omega_1 = V^*(\Omega) \). Then by Lemma 4.2,

\[
V^*(V(\Omega_1)) \cong \Omega_1 = V^*(\Omega).
\]

So \( V(V(\Omega_1)) \cong V(\Omega) \). Again by Lemma 4.2,

\[
V(\Omega_1) \cong V^*(V(V(\Omega_1))) \cong V^*(V(\Omega)) \cong \Omega.
\]

If \( S \) is a finitely generated semigroup and if \( \Omega \) is the maximal semilattice image of \( S \), then clearly \( \Omega \) is finite and \( (X(S), \subseteq) \cong (X(\Omega), \subseteq) \). By Theorem 3.5, Lemma 4.2, Corollaries 4.3, 4.4, we have.

**Theorem 4.5.** (1) Let \( (L, \vee, \wedge) \) be a finite lattice. Then \( L \cong X(\mathcal{P}) \) for some rational polytope \( \mathcal{P} \) if and only if \( \Omega = V^*(L, \wedge) \) is a semilattice and \( \Omega \) is
isomorphic to the maximal semilattice image of some finitely generated, commutative, idempotent free, totally cancellative semigroup.

(2) Let $\Omega$ be a finite semilattice. Then $\Omega$ is the maximal semilattice image of some finitely generated, commutative, idempotent free, totally cancellative semigroup if and only if $(X(\Omega), \mathcal{C})$ is isomorphic to $(X(P), \mathcal{C})$ for some rational polytope $P$.

If $P$ is a polytope, call $X(P)$, the face lattice of $P$. By a theorem of Tarski (see [4; p. 91]), the enumeration problem for face lattices of polytopes is solvable. However, for rational polytopes the problem is not yet solved [4; p. 92]. By Theorem 4.5, we have,

**Theorem 4.6.** The enumeration problem for face lattices of rational polytopes is solvable if and only if the enumeration problem for maximal semilattice images of finitely generated, commutative, idempotent-free totally cancellative semigroups, is solvable.

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