RESEARCH NOTES

A CHARACTERIZATION OF PSEUDOCOMPACTNESS

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ABSTRACT. It is proved here that a completely regular Hausdorff space X is
pseudocompact if and only if for any continuous function f from X to a pseudo-
compact space (or a compact space) Y, \( f^\# \) is a z-ultrafilter whenever \( \# \) is a
z-ultrafilter on X.

KEY WORDS AND PHRASES. Pseudocompact, \( \beta X \), z-filter, z-ultra function.

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1. INTRODUCTION.

For notations and basic results one is referred to [1]. We only consider
here completely regular Hausdorff spaces.

Let \( f \) be continuous from \( X \) to \( Y \). Let \( \# \) be a z-ultrafilter on \( X \), then \( f^\# \)
denotes the z-filter \( \{ B \in Z(Y): f^{-1}(B) \in \# \} \) on \( Y \) and is known to be prime. We
further know that a prime z-filter is contained in a unique z-ultrafilter. Let
\( \Delta(f)\# \) denote the z-ultrafilter containing \( f^\# \). Thus we have a function \( \Delta(f) \)
from \( \beta X \) to \( \beta Y \) sending \( \# \) to \( \Delta(f)\# \). The function \( f \) is called z-ultra if \( f^\# = \Delta(f)\# \)
for every z-ultrafilter \( \# \) on \( X \).
2. MAIN RESULTS

**PROPOSITION.** A continuous function $f$ from $X$ to $Y$ is $z$-ultra if and only if for every zero-set $B$ in $Y$, $\Delta(f)^{-1}(\overline{B}^Y) = f^{-1}(B)$.

**PROOF.** Let $f$ be $z$-ultra. Then, $\phi \in \Delta(f)^{-1}(\overline{B}^Y)$ if and only if $\Delta(f)\phi = f^* \in \overline{B}^Y$. But this is equivalent to $B \in f^* \phi$ or to $f^{-1}(B) \in \phi$, which happens if and only if $\phi \in f^{-1}(B)$.

Conversely, $B \in f^* \phi$ if and only if $\phi \in f^{-1}(B)$, i.e. $\Delta(f)\phi \in \overline{B}^Y$, since $f^{-1}(B) = \Delta(f)^{-1}(\overline{B}^Y)$. But $\Delta(f)\phi \in \overline{B}^Y$ is equivalent to saying that $B \in \Delta(f)\phi$.

We see that $f^* = \Delta(f)\phi$.

In order to prove the main theorem of the paper we need the following observations for pseudocompact spaces. If $X$ is pseudocompact, then a subset of $\beta X$ is a zero-set if and only if it is closure of a zero-set in $X$ and conversely, a subset of $X$ is a zero-set in $X$ if and only if its closure is so in $\beta X$.

**THEOREM.** If a space $X$ is pseudocompact then any continuous function $f$ from $X$ to any pseudocompact space $Y$ is $z$-ultra. Conversely, if the inclusion of $X$ in $\beta X$ is $z$-ultra, then $X$ is pseudocompact.

**PROOF.** Let $B$ be a zero-set in $Y$. Since $\overline{B}^Y$ is a zero-set in $\beta Y$ as $Y$ is pseudocompact, $\Delta(f)^{-1}(\overline{B}^Y)$ is a zero-set in $\beta X$. Pseudocompactness of $X$ implies that $\Delta(f)^{-1}(\overline{B}^Y) = A^{\beta X}$ for some zero-set $A$ in $X$. We show that $A = f^{-1}(B)$.

Since $\Delta(f)/X = f$, we observe that $\Delta(f)^{-1}(B) \cap X = f^{-1}(B)$. Clearly, $\Delta(f)^{-1}(\overline{B}^Y) \cap X = \Delta(f)^{-1}(B) \cap X = f^{-1}(B)$. Next, $\Delta(f)^{-1}(\overline{B}^Y) \cap X = A^{\beta X} \cap X = A$. Hence $f^{-1}(B) = A$, and we have $f$ to be $z$-ultra.

Conversely, let $i$ be the inclusion of $X$ in $\beta X$. Since $\Delta(i)/X = i$, $\Delta(i)$ is the identity on $\beta X$. Let $B$ be a nonempty zero-set in $\beta X$. Since $i$ is $z$-ultra, $\overline{B}^X \cap X = \Delta(i)^{-1}(B) = i^{-1}(B) = B \cap X$ and [1,61.1] shows that $X$ is pseudocompact.

As an application of our theorem we prove the following well known theorem due to Glicksberg [2].
THEOREM. If $X$ is pseudocompact and $Y$ is compact, then $X \times Y$ is pseudocompact.

PROOF. Let $f: X \times Y \to Z$ be a continuous function, $Z$ some pseudocompact space. Consider a $z$-ultrafilter $\phi$ on $X \times Y$. Let $\pi_2: X \times Y \to Y$ denote the projection on the second coordinate. Since $Y$ is compact and $\pi_2^* \phi$ is a $z$-filter, it is fixed as well. Let $y_0 \in \bigcap \pi_2^* \phi$. Hence $\phi_1$, the restriction of $\phi$ to the subspace $X \times \{y_0\}$ is a $z$-ultrafilter on $X \times \{y_0\}$. Let $f_1$ denote the restriction of $f$ to the subspace $X \times \{y_0\}$. Since $X$ is pseudocompact, $f_1$ is $z$-ultra. Clearly, $f^* \phi \subseteq f_1^* \phi_1$. Next, let $B \in f_1^* \phi_1$. Hence $f_1^{-1}(B) \in \phi_1$. Since $f_1^{-1}(B)$ contains $f_1^{-1}(B)$, $f^{-1}(B)$ intersects every member of $\phi$. Thus $f^{-1}(B) \in \phi$ as it is a $z$-ultrafilter. We get that $B \in f^* \phi$. Hence $f^* \phi = f_1^* \phi_1$ and it follows that $f$ is $z$-ultra.

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