ON RANK 4 PROJECTIVE PLANES

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ABSTRACT. Let a finite projective plane be called rank m plane if it admits a collineation group G of rank m, let it be called strong rank m plane if moreover \( G_P = G_1 \) for some point-line pair (P,1). It is well known that every rank 2 plane is desarguesian (Theorem of Ostrom and Wagner). It is conjectured that the only rank 3 plane is the plane of order 2. By [1] and [7] the only strong rank 3 plane is the plane of order 2. In this paper it is proved that no strong rank 4 plane exists.

KEY WORDS AND PHRASES. Projective planes, rank 4 groups.


1. INTRODUCTION.

In [6] Kallaher gives restrictions for the order n of a finite rank 3 pro-
jective plane and conjectures that no such plane exists if \( n \neq 2 \). Let a finite projective plane be called a strong rank \( m \) projective plane if it admits a rank \( m \) collineation group \( G \) such that \( G_P = G_1 \) for some point-line pair \((P,1)\). By Bachmann [1] and Kantor [7] no strong rank 3 projective plane of order \( n \neq 2 \) exists. If the conjecture is true that for projective designs the representations on the points and on the blocks of an arbitrary transitive collineation group are similar (see Dembowski [2], p. 78), then every rank \( m \) projective plane is a strong rank \( m \) plane.

We shall prove in this article the following

**THEOREM:** No strong rank 4 projective plane exists.

To prove the Theorem we first divide the strong rank 4 planes into 3 classes (see Lemma 2 and 3). Then we associate with each such plane \((0,1)\)-matrices \( A \) and \( C \) of trace 0 (see [3]). Finally we show that for each class the trace condition contradicts the integrality of the multiplicities of the eigenvalues of \( A \) or \( C \).

We shall use the following notations, definitions and basic results (see Dembowski [2]):

A collineation group of a projective plane has equally many point orbits and line orbits. The rank of a transitive permutation group is the number of orbits of the stabilizer of one of the permuted elements. If \( G \) is a (point or line) transitive collineation group of a projective plane, then the point and line ranks are equal (Kantor [8]). A rank \( m \) projective plane is a projective plane which admits a transitive collineation group whose (point or line) rank is \( m \) \((m \geq 2)\). The lines (points) are identified with the set of points (lines) on them. We write \( P \in \Gamma^G \) if and only if \( P \in \Gamma^\gamma \) for all \( \gamma \in G \).
2. **Proof of the Theorem.**

Let \( \mathcal{P} = (P, L, \xi) \) be a projective plane of finite order \( n \) and let \( G \) be a rank 4 collineation group of \( \mathcal{P} \) such that \( G_{P_0} = G_{1_0} \) for some point-line pair \((P_0, 1_0)\). It is easily seen that \( n \geq 3 \). A bijective map \( \sigma : P \rightarrow L \) is defined by \( P^\sigma = 1 \) if and only if \( P = P_0^\gamma \) and \( 1 = 1_0^\gamma \) for some \( \gamma \in G \). If \( i \in \mathbb{N} \) we write \( i \) for \( P_i^\sigma \). Clearly \( P_0^\sigma = 1_0 \) and

\[
P_0^{\gamma \sigma} = P_0^{\gamma} \quad \text{and} \quad 1_0^{\sigma^{-1} \gamma} = 1_0^{\gamma^{-1}} \text{ for all } P \in \mathcal{P}, 1 \in L, \gamma \in G. \tag{1}
\]

For \( P \in \mathcal{P} \), \( G_P \) has exactly 4 orbits \( \{P\}, \Delta(P), \Gamma(P), \Pi(P) \). We choose the notation in such a way that

\[
(\Delta(P))^{\gamma} = \Delta(P_0^{\gamma}), \quad (\Gamma(P))^{\gamma} = \Gamma(P_0^{\gamma}), \quad (\Pi(P))^{\gamma} = \Pi(P_0^{\gamma}) \text{ for all } P \in \mathcal{P}, \gamma \in G. \tag{2}
\]

**Lemma 2.1:** If \( A, A', A'' \in \{\Delta, \Gamma, \Pi\} \), then \( |\bigwedge_1(A) \cap \bigwedge_2(B)| = |\bigwedge_1(A') \cap \bigwedge_2(B')| \) if \( A \in \bigwedge_3(B) \) and \( A' \in \bigwedge_3(B') \).

**Proof:** If \( A \in \bigwedge_3(B), A' \in \bigwedge_3(B'), \) then for some \( \gamma \in G, \gamma_0 \in G_B \)

\[
B' = B^{\gamma} = B_0^{\gamma \gamma_0}, \quad A' = A_0^{\gamma \gamma_0}, \quad \text{whence by (2)}
\]

\[
|\bigwedge_1(A') \cap \bigwedge_2(B')| = |\bigwedge_1(A_0^{\gamma \gamma_0}) \cap \bigwedge_2(B_0^{\gamma \gamma_0})| = |\bigwedge_1(A) \cap \bigwedge_2(B)| ^{\gamma \gamma_0} = |\bigwedge_1(A) \cap \bigwedge_2(B)| .
\]

**Lemma 2.2:** Suppose that \( P_0 \in 1_0 \). Then \( 1_0 - \{P_0\} \) and \( G_{P_0} - 1_0 \) are \( G_{P_0} \)-orbits, say \( \Delta(P_0) = 1_0 - \{P_0\} \) and \( \Gamma(P_0) = G_{P_0} - 1_0 \) with \( P_2 = \Gamma(P_0) \). \( P_1 \in \Delta(P_0) \) and \( P_3 \in \Pi(P_0) \) can be chosen such that \( P_1 \in 1_0; P_0, P_2, P_3 \in 1_2; P_2 \in 1_1; P_1 \notin 1_3 \) (Fig. 1).

The case described by Lemma 2 will be called case I.

**Proof:** If \( 1_0 - \{P_0\} \) is not a \( G_{P_0} \)-orbit, then it is the union of 2 orbits, say \( 1_0 - \{P_0\} = \Delta(P_0) \cup \Gamma(P_0) \). Then \( P_0 - 1_0 \) is a line orbit \( P_0 \) and \( \Pi(P_0) = G_{P_0} \) with \( 1 = P_0 \). This leads to the contradiction.
\[ n = |G_{P_0}| = |P \times G_{P_0}| = |P \times G_{P_0}| = |P \times G_{P_0}| = |P - 1_o| = n^2. \]

Hence we may assume that \( \Delta(P_o) = 1_o - \{P_o\} \).

Dually: \( P_o - \{1_o\} \) is a \( G_{P_0} \)-orbit, say \( 1_o = P_o - \{1_o\} \) where \( P_2 = \Gamma(P_o) \) (note that \( P_o \notin 1_o \)).

\[ |G_{P_0}| = |1_o| = |P_o - \{1_o\}| = n \text{ implies } \Gamma(P_o) \cap 1_o = \{P_2\}. \]

For any point \( Q \neq P_0 \), \( P_2 \) on \( 1_o \) holds \( Q^{P_0} = \Pi(P_o) \).

Let \( P_1 \in \Delta(P_o) \). If \( P_2 \notin 1_o \) put \( P_1 = P_2 \). If \( P_2 \in 1_o \) then

\[ |G_{P_0}, P_2| = |P_1, G_{P_0}, P_2| \geq |(1_o \cap 1_o)| = n - 1 = |(1_o \cap 1_o)| = n. \]

It follows that

\[ G_{P_0}, P_2 = P_1 \text{ for some point } \]

\( P_1 \in \Delta(P_o) \) and hence \( P_2 \in 1_o \).

It remains to prove that \( P_3 \in \Pi(P_o) \cap 1_o \) exists such that \( P_1 \notin 1_o \). If no such \( P_3 \) exists then \( P_1 \notin Q^{P_0} \) for all \( Q \in 1_o - \{P_o, P_2\} \) and hence \( G_{P_0}, P_2 \neq G_{P_0}, P_1 \).

Let \( \gamma \in G \) be such that \( P_2 = P_0 \). Then \( 1_o = \gamma_0 \) and therefore \( \gamma \in 1_o, P_o \neq P_0 \) and \( P_1 = P_1 \) for some \( \gamma \in G_{P_0} \). It follows that \( G_{P_0}, P_1 = (\gamma_0)^{-1} G_{P_0}, P_2 \gamma \).

Hence

\[ G_{P_0}, P_1 = G_{P_0}, P_2. \] (3)

Further

\[ P_2 \notin 1_o \text{ for some } \gamma \in G_{P_0}, \] (4)

for otherwise \( P_2 = 1_o \) for all \( \gamma_0, \gamma \in G_{P_0} \) which cannot occur.

\[ P_2 \in 1_o \text{ for some } \gamma \in G_{P_0} \text{ if and only if } \gamma \in G_{P_2}. \] (5)

To prove (5) note that by (4) through any point of \( 1_o - \{P_o\} \) goes at least \( G_{P_0} \) (3) and \( P_2 \in 1_o \) then imply (5).

Let's apply (5) to \( G_{P_1} \) in place of \( G_{P_0} \):

\[ \Delta(P_1) = 1_o - \{P_2\}; \Gamma(P_1) = \Gamma(P_0) \gamma = G_{P_0} \gamma^{-1} G_{P_0} \gamma = G_{P_1}. \]
where \( \gamma \in G \) such that \( P_0^\gamma = P_1, P_2^\gamma = P_0 \); \( \Pi(P_1) = S P_1 \) for some \( S \in l_0 \) - \( \{P_0, P_1\} \); hence \( P_0^\gamma \in l_2 \) for some \( \gamma_1 \in G P_1 \) if and only if \( \gamma_1 \in G P_0 \).

It follows that \( R \notin P_0 P_1 \) for any \( R \in l_2 - \{P_0, P_2\} \). Let \( R = R' \) for some such \( R \).

Of the 3 orbits \( (P_0, l_1)^G \), \( (P_0, l_2)^G \), \( (P_0, r)^G \) induced by \( G \) on \( P \times \perp - (P_0, l_1)^G \) only one consists of flags. Thus \( (P_1, l_0) \) and \( (P_1, r) \) and then also \( (P_0, l_1) \) and \( (R, l_1) \) belong to the same \( G \)-orbit. This contradicts \( R \notin P_0 P_1 \).

Hence there exists \( P_3 \in \Pi(P_0) \cap l_2 \) such that \( P_1 \notin l_3 \).

**Lemma 2.3:** Suppose that \( P_0 \notin l_0 \). Then \( l_0 \) and dually \( P_0 \) are \( G P_0 \)-orbits, say \( \Delta(P_0) = l_0 \). \( P_1 \in \Delta(P_0) \), \( P_2 \in \Gamma(P_0) \), \( P_3 \in \Pi(P_0) \) can be chosen such that either \( P_0, P_2, P_3 \in l_1 \); \( P_1 \in l_2, l_3 \); \( P_2 \notin l_3 \); \( P_3 \notin l_2 \) or \( P_0, P_1, P_2 \in l_2 \); \( P_1 \notin l_0 \); \( \Gamma(P_0) \cap l_2 = \{P_2^\gamma\} \) for some \( \gamma \in G P_0 \); \( P_2^\gamma \in l_1 \); \( P_1, P_2, P_3 \notin l_1, l_3 \). In both cases \( n \geq 4 \).

The 2 cases described by Lemma 3 will be called case III resp. case II2 (Fig. 2).

**Proof:** It is easily seen that \( l_0 \) and \( P_0 \) are \( G P_0 \)-orbits; say \( \Delta(P_0) = l_0 \). Let \( P_1 \in \Delta(P_0) \). We have to distinguish 2 cases:

Case III: \( P_0 \notin l_1 \)

Case II2: \( P_0 \notin l_1 \).

**Case II1:** Clearly \( P_0 = P_1 \) and \( \Gamma(P_0) = P_2 \); \( \Pi(P_0) = P_3 \) for some \( P_2, P_3 \in l_1 - \{P_0, l_0 \cap l_1\} \). If \( P_2 \in l_3 \) then \( (P_2, l_3) \in (P_0, l_1)^G \), hence \( (P_3, l_2) \in (P_1, l_0)^G \), so \( P_3 \in l_2 \).
Analogously $P_2 \in l_3$ if $P_3 \in l_2$. Thus

$$P_2 \in l_3 \text{ if and only if } P_3 \in l_2. \quad (6)$$

Similarly one proves

$$P_1 \in l_2, l_3. \quad (7)$$

If $n > 3$ then, by (6), we can choose $P_2, P_3$ such that $P_2 \notin l_3, P_3 \notin l_2$.

Let's show that $n > 3$ (Fig. 3). Suppose that $n = 3$. Put $P_4 = l_\circ \cap l_1$.

Then, since $P_\circ \in l_1$ and $P_1 \in l_\circ$,

$$l_4 = P_\circ P_1. \text{ Let } P_5 \in l_\circ - \{P_1, P_4\}.$$

Then $P_\circ \in l_5$ and then $l_5 \cap l_2 = (P_\circ P_5 \cap l_3)P_4 \cap l_2$. Denote this point by $T$. Clearly $P_2P_5 \cap l_2 = T$. Since $(P_2P_5)^{d-1} \in l_2 \cap l_5$ we obtain the contradiction

$$(P_2P_5)^{d-1} \notin P_2P_5.$$
CASE II: We may assume that \( P_0, P_1, P_2 \in L \) where \( P_2 \in \Gamma(P_0) \). Then

\[ G_{P_0, P_1} = G_{P_0, P_2} \].

We first assume that \( n > 3 \). Let \( l_2 \cap \Gamma(P_0) = \{ \gamma_0 \} \) with some \( \gamma_0 \in G_{P_0} \). Then \( P_3 \gamma_0 \in l_2 - \{ P_0, P_1, P_2 \} \) and hence, since \( P_2 \gamma_0 \) is invariant under \( G_{P_0, P_1} \) and since \( n > 3 \), \( l_2 \cap l_2 = P_0 \).

The only \( G \)-orbit of \( P \times L \) consisting of flags is \( (P_0, l_2)^G \). Hence \( P_1, l_2 \), \( (P_2, l_2) \), \( (P_0, l_0) \in (P_0, l_2)^G \). \( P_0 \notin l_1 \) then implies that \( (P_2, l_1), (P_2, l_3) \), \( (P_0, l_1), (P_2, l_0) \notin (P_0, l_2)^G \), in particular \( P_2 \notin l_0, l_1, l_3 \).

If \( P_1 \notin l_3 \) then \( (P_1, l_3) \notin (P_0, l_2)^G \) and hence \( (P_3, l_1) \notin (P_2, l_0)^G \).

Since also \( (P_0, l_1) \notin (P_2, l_0)^G \) we have \( P_0 \gamma_1 = P_3 \) for some \( \gamma_1 \in G_{l_1} = G_{P_1} \).

This implies that \( G_{P_1} \) is transitive on \( l_2 - \{ P_1, P_2 \} \) which is impossible.

Hence \( P_1 \notin l_3 \).

If \( n = 3 \) then \( l_2 = \{ P_0, P_1, P_2, P_3 \} \). \( \gamma_0 \) is of order 4, for if \( \gamma_0^2 = 1 \) then \( (P_2, l_2^2) \notin (P_2, l_2)^G \) which is impossible. Moreover \( P_2 \neq p_2 \) since otherwise \( \gamma_0^2 \in G_{P_0, P_2} = 1 \). It follows that \( |(P_2^2 P_2^2) G_{P_0}| = 1 \) which contradicts

\[ (P_2^2 P_2^2) = \{ P_2^2 P_2^0, P_2^2 P_2^0 \} \].

This completes the proof of the Lemma.

Let us now associate with \( (G, P) \) 3 \((0,1)\)-matrices.

If \( \rho (P) \) is a \( G \)-orbit then let \( \rho' (P) \) denote the paired orbit (see Wielandt [9]). If \( Q \in \rho (P) \) then \( Q = P^\gamma \) for some \( \gamma \in G \) and \( \gamma \in (\rho (P))^\gamma = \rho (P^\gamma) = \rho (Q) \).

Hence \( Q = P \in \rho' (Q) \), i.e.

\[ Q \in \rho (P) \] implies that \( P \in \rho' (Q) \).
This implies that in

**Case I:**
\[ A'(P) = A(P), \quad r'(r) = r(P) \]

**Case III:**
\[ A'(P) = A(P), \quad r'(r) = r(P) \]

**Case II:**
\[ A'(P) = A(P), \quad r'(r) = r(P) \]

Now let \( P = \{ P_1, P_2, \ldots, P_v \} \), \( L = \{ l_1, l_2, \ldots, l_v \} \), \( l_k = P_k^o \) (\( k = 1, 2, \ldots, v \)). Let \( A \) be the \((0,1)\)-matrix with rows enumerated by the points \( P_k \) and columns by \( A(P_k) \) and such that \( \langle P_k, A(P_l) \rangle = 1 \) if and only if \( P_k \in A(P_l) \). Let \( B, C \) be the analogous matrices with \( R(P_k) \) resp. \( \Pi(P_k) \) in place of \( A(P_k) \).

We have in

**case I:**
\[ A^t = B, \quad C^t = C \]

**case III:**
\[ A^t = A, \quad B^t = C \]

Let \( k = |A(P)|, \quad l = |R(P)|, \quad m = |\Pi(P)|, \quad n = |\Gamma(P)| \).

\[ |A(P) \cap A(Q)| = \begin{cases} \lambda & \text{if } Q \in \Delta(P) \\ \mu & \text{if } Q \in \Pi(P) \end{cases} \]

\[ |\Pi(P) \cap \Pi(Q)| = \begin{cases} \lambda' & \text{if } Q \in \Delta(P) \\ \mu' & \text{if } Q \in \Pi(P) \end{cases} \]

A straightforward calculation shows that

\[ I + A + B + C = J \], the \( v \times v \)-matrix with 1's in every entry

\[ A^t A = k I + \lambda A + \mu B + \nu C \]

\[ C^t C = m I + \mu' A + \nu' B + \lambda' C \]
Now we determine the eigenvalues of $A$ in case III and of $C$ in the cases I and II.

**CASE III:** $k = n + 1$

\[ l = n_2(n + 1) \quad \text{where} \quad n_2 = |P_2^{P_0, P_1}| \]

\[ m = n_3(n + 1) \quad \text{where} \quad n_3 = |P_3^{P_0, P_1}| \]

\[ k + 1 + m + 1 = v = n^2 + n + 1, \quad n_2 + n_3 = n - 1, \quad \lambda = \mu = \nu = 1. \]

It follows that $A^2 = A^TA = (n + 1)I + A + B + C = nI + J$; hence

\[(A - (n + 1)I)(A^2 - nI) = 0. \]

This gives the eigenvalues of $A$:

\[ \lambda_1 = n + 1, \quad \lambda_{2,3} = \pm \sqrt{n}. \]

**CASE I:** $k = 1 = n, \quad m = n(n - 1), \quad k + 1 + m + 1 = v = n^2 + n + 1.$

We have

\[ \lambda' = |\mathbb{I}(P_0) \cap \mathbb{I}(P_3)| \]

\[ \mu' = |\mathbb{I}(P_0) \cap \mathbb{I}(P_1)| \]

\[ \nu' = |\mathbb{I}(P_0) \cap \mathbb{I}(P_2)|. \]

Let's calculate $\lambda'$:

\[ n(n - 1) = |\Pi(P_3)| = |\Pi(P_3) \cap \Delta(P_0)| + |\Pi(P_3) \cap \Gamma(P_0)| + |\Pi(P_3) \cap \Pi(P_0)| + 1 \quad (9) \]

(note that $\Gamma(P_3) = P_2^{P_3}$ and hence $P_0 \in \Pi(P_3)$).

\[ n = |\Delta(P_0)| = |\Delta(P_0) \cap \Delta(P_3)| + |\Delta(P_0) \cap \Gamma(P_3)| + |\Delta(P_0) \cap \Pi(P_3)|. \quad (10) \]

Clearly

\[ |\Delta(P_0) \cap \Delta(P_3)| = 1 \quad (11) \]

\[ |\Delta(P_0) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_0)| = 2. \quad (12) \]
PROOF of (12): \( P_o \in \Pi(P_3) \) and \( P_3 \in \Pi(P_o) \), hence, by Lemma 1, \(|\Delta(P_o) \cap \Gamma(P_3)| = \frac{1}{n - 1} \) implies that \(|Y_o \cap \Gamma(P_3)| = n - 1\). Hence \(|Y_o \cap \Gamma(P_3)| = n - 1\). Since \( P_1 \notin \gamma_o \) we then have \( P_1 \gamma_o P_2 = P_1 \), i.e. \( G_{P_o} P_2 \leq G_{P_o} P_1 \). Since both groups are conjugate (see the proof of Lemma 2) this gives \( G_{P_o} P_1 = G_{P_o} P_2 \).

Moreover \( P_2 \in \gamma^* \) if and only if \( \gamma_o \in G_{P_1} \). Thus \( P_2 \in \gamma_1 \), \( P_2 \in \gamma_1 \), \( P_2 \in \gamma_1 \), \( P_1 \not\in \gamma_1 \) for some \( \gamma' \), \( \gamma'' \), \( \gamma'' \), \( \gamma'' \). Since, by the above, \( G_{P_o} P_2 \) is transitive on \( 1_0 \) - \( \{P_o, P_1\} \), \( |P_2 \gamma_o P_2 \cap P_o| = n - 1 \); hence \( |1_0 \cap P_2 P_1| = |1_0 \cap P_2 P_1| = 2 \). This proves (12).

Equations (10), (11), (12) imply

\[ |\Pi(P_3) \cap \Delta(P_o)| = n - 3. \tag{13} \]

To determine \(|\Pi(P_3) \cap \Gamma(P_o)|\) we use

\[ n = |\Gamma(P_o)| = |\Gamma(P_o) \cap \Delta(P_3)| + |\Gamma(P_o) \cap \Gamma(P_3)| + |\Gamma(P_o) \cap \Pi(P_3)|. \tag{14} \]

By (12) \( |\Gamma(P_o) \cap \Delta(P_3)| = 2 \). Since \( \Gamma(P_3) = \gamma_{P_3} \), \( |\Gamma(P_o) \cap \Gamma(P_3)| = \gamma_{P_3} \gamma_{P_3} \), \( |\gamma_{P_3} \cap P_2 P_3| = |1_2 \cap 1_2 P_3| = 1 \). It follows that

\[ |\Pi(P_3) \cap \Gamma(P_o)| = n - 3. \tag{15} \]

Equations (9), (13) and (15) imply that

\[ \lambda' = n^2 - 3n + 5. \tag{16} \]

Analogously we calculate \( \mu' \) and \( \nu' \):

\[ n(n - 1) = |\Pi(P_1) \cap \Delta(P_o)| + |\Pi(P_1) \cap \Gamma(P_o)| + |\Pi(P_1) \cap \Pi(P_o)| \]

with \(|\Pi(P_1) \cap \Delta(P_o)| = n - 1\).

In

\[ n = |\Gamma(P_o)| = |\Gamma(P_o) \cap \Delta(P_1)| + |\Gamma(P_o) \cap \Gamma(P_1)| + |\Gamma(P_o) \cap \Pi(P_1)| \]
\[ |\Gamma(P_o) \cap \Delta(P_1)| = 1 \] by the proof of (12) and \[ |\Gamma(P_o) \cap \Gamma(P_1)| = G_{P_o}^G \cap G_{P_1}^G = |1_2 o \cap 1_o^2| = 0. \] Hence \[ |\Pi(P_1) \cap \Gamma(P_o)| = n - 1 \] and thus
\[ \mu' = (n - 1)(n - 2). \] (17)

\[ n(n - 1) = |\Pi(P_2)| = |\Pi(P_2) \cap \Delta(P_o)| + |\Pi(P_2) \cap \Gamma(P_o)| + |\Pi(P_2) \cap \Pi(P_o)|. \]

In\[ n = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_2)| + |\Delta(P_o) \cap \Gamma(P_2)| + |\Delta(P_o) \cap \Pi(P_2)| \]
\[ |\Delta(P_o) \cap \Delta(P_2)| = 0 \text{ and } |\Delta(P_o) \cap \Gamma(P_2)| = G_{P_o}^G \cap G_{P_2}^G \]
\[ = |1_1 o \cap 1_1^2| = 1 \]
(note that \[ |1_1 o \cap 1_2| = |1_1^2| = n \] and hence \[ \Gamma(P_2) = P_1 o 2 \]). Hence \[ |\Pi(P_2) \cap \Delta(P_o)| = n - 1. \]

Further \[ |\Pi(P_2) \cap \Gamma(P_o)| = |\Gamma(P_o)| - |\Gamma(P_2) \cap \Delta(P_2)| - |\Gamma(P_2) \cap \Gamma(P_2)| - 1 \]
where \[ |\Gamma(P_o)| = n, |\Gamma(P_2) \cap \Delta(P_2)| = 0 \] and \[ |\Gamma(P_2) \cap \Gamma(P_2)| = |G_{P_2}^G \cap G_{P_2}^G| = |1_2 o \cap 1_2^2| = 0. \] Hence \[ |\Pi(P_2) \cap \Gamma(P_o)| = n - 1. \] It follows that
\[ \nu' = (n - 1)(n - 2). \] (18)

Equations (16), (17) and (18) imply that \[ c^2 = c c = n(n - 1) I + \]
\[ (n - 1)(n - 2)(A + B) + (n^2 - 3n + 5) C = n(n - 1) I + (n - 1)(n - 2)(I - I) + 3C \]
and then \[ (C - n(n - 1) I)(c^2 - 3C - 2(n - 1) I) = 0. \]

The eigenvalues of \( C \) are \[ \lambda_1 = n(n - 1); \lambda_{2,3} = (3 \pm \sqrt{8n - 1})/2. \]

CASE II2: \( k = 1 = n + 1, m = (n - 2)(n + 1), k + 1 + m + 1 = v = n^2 + n + 1. \)

By the proof of Lemma 3 \( n \geq 4. \) Let's determine \( \lambda', \mu', \nu' \):
\[ (n + 1)(n - 2) = |\Xi(P_3)| = |\Xi(P_3) \cap \Delta(P_o)| + |\Xi(P_3) \cap \Gamma(P_o)| + |\Xi(P_3) \cap \Pi(P_o)| - 1. \] In \( n + 1 = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_3)| + |\Delta(P_o) \cap \Gamma(P_3)| + |\Delta(P_o) \cap \Pi(P_3)| \]
\[ \text{clearly } |\Delta(P_o) \cap \Delta(P_3)| = 1. \] Let's show that
\[ |\Delta(P_1) \cap \Gamma(P_o)| = 2 \] (19)
\[ |\Delta(P_1) \cap \Gamma(P_o)| = 2. \] (20)
PROOF of (19) and (20): By Lemma 1 \(|\Delta(P_o) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_o)|\).

For \(\gamma'_o \in G_{P_o}\)

\[
\gamma'_o \in \gamma'_o G_{P_o}, 1_1 \neq p_2 o 2 \text{ if and only if } \gamma'_o \in G_{P_o} P_1,
\]

(21)

for otherwise, \(\gamma'_o 1_1 o P_1, 1_1 \neq p_2 o 2, |1_1 o G_{P_o} P_1| = n - 2, p_2 \in
\gamma'_o G_{P_o} P_1\), which leads to the contradiction \(n + 1 = |1_1 o | > (|P_2| - 2) + (|P_2| - 2) = 2(n - 1)\).

Further

\[
\gamma'_o \notin 1_1 \cup \gamma'_o P_2 \text{ for some } \gamma'_o \in G_{P_o}.
\]

otherwise, since \(|P_2 o | > 5\), every line of \(1_1 o G_{P_o}\) would contain at least 3 points of \(P_2 o\) and this would imply that a point of \(P_2 o - (1_1 \cup \gamma'_o P_2)\) exists. By (21) and (22) \(\gamma'_o G_{P_o} P_2, 1_3 o P_2, 1_2 o \neq p_2 o 2 \text{ for some } \gamma'_o \in G_{P_o}\). Since \(|1_3 o G_{P_o} P_1| = n - 2, |P_2 o \cap 1_3| = |\Gamma(P_o) \cap \Delta(P_3)| = 2\). This proves (19).

Each of the \(n - 2\) lines of \(1_3 o - \{\gamma'_o P_2\}\) through \(\gamma'_o\) contains exactly one point of \(P_2 o - \{\gamma'_o\}\). Together with \(P_2 o, P_2\) this gives \(n\) points of \(G_{P_o}\). It follows that exactly one point of \(P_2 o - \{\gamma'_o\}\) lies on \(1_3\). This proves (20).

By means of (19) we obtain

\[
|\Pi(P_3) \cap \Delta(P_o)| = n - 2.
\]

In 

\[
|\Gamma(P_o)| = |\Gamma(P_o) \cap \Delta(P_3)| + |\Gamma(P_o) \cap \Gamma(P_3)| + |\Gamma(P_o) \cap \Pi(P_3)|
\]

\[
|\Gamma(P_o) \cap \Delta(P_3)| = 2 \text{ by (19) and } |\Gamma(P_o) \cap \Gamma(P_3)| = |P_2 o \cap P_2 o 3| = |1_2 o \cap 1_2 o 3| = 1\).
\]

Hence \(|\Pi(P_3) \cap \Gamma(P_o)| = n - 2\). It follows that \(\lambda' = n^2 - 3n + 1\).

\[
u' = |\Pi(P_o) \cap \Pi(P_1)| = |\Pi(P_1)| - |\Pi(P_1) \cap \Delta(P_o)| - |\Pi(P_1) \cap \Gamma(P_o)|\]

where \(|\Pi(P_1)| = (n + 1)(n - 2), |\Pi(P_1) \cap \Delta(P_o)| = n - 2\) and \(|\Pi(P_1) \cap \Gamma(P_o)| = |\Gamma(P_o)| - |\Gamma(P_o) \cap \Delta(P_1)| - |\Gamma(P_o) \cap \Gamma(P_1)| - |\Gamma(P_3) \cap \Delta(P_1)| = 2 \text{ by (20) and } |\Gamma(P_o) \cap \Gamma(P_1)| = |P_2 o \cap P_2 o 1| = |1_2 o \cap 1_2 o 1| = 1\).

Hence \(|\Pi(P_1) \cap \Gamma(P_o)| = n - 2\) and \(\nu' =...
(n - 2)(n - 1).

v' = |\Pi(P_o) \cap \Pi(P_2)| = |\Pi(P_2)| - |\Pi(P_2) \cap \Delta(P_o)| - |\Pi(P_2) \cap \Gamma(P_o)|

where

|\Pi(P_2)| = (n + 1)(n - 2),

|\Pi(P_2) \cap \Delta(P_o)| = |\Pi(P_2) \cap \Delta(P_1)|

by Lemma 1

= |\Delta(P_1)| - |\Delta(P_1) \cap \Delta(P_o)| - |\Delta(P_1) \cap \Gamma(P_o)|

= (n + 1) - 1 = n - 2

by (20),

|\Pi(P_2) \cap \Gamma(P_o)| = |\Pi(P_2) \cap \Gamma(P_1)|

by Lemma 1

= |P_3 \cap P_1| = \left| \left\{ g \in G : |g_1^o \in G, g_1^o \in G \right\} \right| = n - 2

since through any point on \( P_2 \) goes exactly one line of \( P_1 \) and one of \( P_2 \). Hence

v' = (n - 2)(n - 1).

It follows that

\( C^2 = C \cdot C = (n + 1)(n - 2) I + (n - 1)(n - 2)(A + B) +

(n^2 - 3n + 1) C = (n + 1)(n - 2) I + (n^2 - 3n + 2)(A + B + C) - C \) and

\( C^2 + C - 2(n - 2) I = (n - 1)(n - 2) J \) whence

\( (C - (n + 1)(n - 2) I)(C^2 + C - 2(n - 2) I) = 0. \)

The eigenvalues of \( C \) are

\( \lambda_1 = (n + 1)(n - 2), \lambda_{2,3} = (-1 \pm \sqrt{8n - 15})/2. \)

REMARK: Let \( \phi : G \rightarrow \text{GL}_v(\mathbb{C}) \) be the matrix representation of \( G \) obtained by

associating with each \( \gamma \in G \) the corresponding permutation matrix \( \phi(\gamma) \) (the ordering

of \( P \) is the same as used in constructing the matrices \( A, B, C \)). By (2) \( \phi(\gamma) \)

commutes with \( A, B, C \) for all \( \gamma \in G \). Hence, by [9] Theorem 28.4, \( \{ I, A, B, C \} \) is

the basis of the commuting algebra \( \mathcal{W}(G) \) of \( \phi \). By [9] Theorem 29.5 \( \mathcal{W}(G) \) is commu-

itative and hence, by [9] Theorem 29.4, the representation \( \phi \) has 4 irreducible

constituents \( D_1 = 1, D_2, D_3, D_4 \), each with multiplicity 1. If \( f_i \) is the degree

of \( D_i \) then \( f_1 = 1 \) and \( \sum_{i=1}^4 f_i = v. \)

Let us finally show how the fact that \( A \) and \( C \) have trace 0 contradicts the

integrality of the multiplicities of \( \lambda_1, \lambda_2, \lambda_3. \)
In the 3 cases \( \lambda_1 \) appears with multiplicity 1. Let \( f \) denote the multiplicity of \( \lambda_2 \); then \( v - f - 1 \) is the multiplicity of \( \lambda_3 \). This leads to

- In case I,
  \[
  0 = n(n - 1) + f(3 + \sqrt{8n + 1})/2 + (n(n + 1) - f)(3 - \sqrt{8n + 1})/2
  \]

- In case II,
  \[
  0 = (n + 1) + f\sqrt{n} + (-\sqrt{n})(n(n + 1) - f)
  \]

- In case III,
  \[
  0 = (n + 1)(n - 2) + f(-1 + \sqrt{8n - 15})/2 + (n(n + 1) - f)(-1 - \sqrt{8n - 15})/2
  \]

In any case this contradicts the fact that \( n \geq 2 \) and \( f \geq 1 \) are integers.

In case III this is clear.

In case I suppose that a prime \( p \) divides \( \sqrt{8n + 1} \). Then \( p \nmid n \), hence \( p \mid 5n + 1 \) and then \( p \mid 3n \), i.e. \( p = 3 \). This implies that \( 8n + 1 = 3^{2i} \) for some \( i \geq 2 \) and that
\[
  n(5n + 1)/\sqrt{8n + 1} = (3^{2i-1})(5 \cdot 3^{2i-1} + 1)/8 \cdot 3^{i-1} \not\in \mathbb{N}.
\]

In case II suppose that a prime \( p \) divides \( \sqrt{8n - 15} \). Then \( p \in \{17, 23\} \) and \( p^2 \nmid n + 1, p^2 \nmid n - 4 \). Hence \( 8n - 15 \in \{17^2, 23^2, 17^2 \cdot 23^2\} \), i.e. \( n \in \{38, 68, 19112\} \). Suppose that \( n = 38 \). Since \( G_{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2} \) is transitive on \( \{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2\} \), \( |G| \) is even. This contradicts the fact that if \( n \equiv 2 \mod 4 \), then the full collineation group is of odd order (Hughes [5]).

Suppose that \( n \in \{68, 19112\} \). Then \( n \) is not a square and \( n^2 + n + 1 \) not a prime.

Hence, since \( G \) is flag-transitive, \( n \) is a prime power (Higman and Mc Laughlin [4]) which is absurd.

REFERENCES


