A GENERALIZATION OF CONTRACTION PRINCIPLE

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ABSTRACT: In this paper, a generalized mean value contraction is introduced. This contraction is an extension of the contractions of earlier researchers and of the generalized mean value non-expansive mapping. Using the generalized mean value contraction, some fixed point theorems are discussed.

KEY WORDS AND PHRASES: Fixed Point, Mean Value Iteration.

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1. INTRODUCTION.

Let T be a self mapping of a Banach space E. The mapping T will be called a generalized mean value contraction mapping if for any $x, y \in E$, there exist non-negative real numbers $a_i$ ($i = 1, 2, \ldots, 5$) such that

$$ \|T_{\lambda}x - TT_{\lambda}y\| \leq a_1 \|x - y\| + a_2 \|x - TT_{\lambda}x\| + a_3 \|y - TT_{\lambda}y\| + a_4 \|x - TT_{\lambda}y\| + a_5 \|y - TT_{\lambda}x\| $$

(1.1)

where $\sum_{i=1}^{5} a_i < 1$ and $T_{\lambda}x = \lambda x + (1-\lambda)Tx$, and $TT_{\lambda}x = T(\lambda x + (1-\lambda)Tx)$, $0 < \lambda \leq 1$ holds.

The contraction (1.1) is more general than the Banach contraction, contractions of
Kannan [1], Chatterjee [2], Hardy and Rogers [3]. When $\lambda=1$ all these contractions follow as a particular case of (1.1), with suitable choice of $a_i$'s. Also, by example, we show that there exist self-mappings which satisfy (1.1), but do not satisfy the well-known contraction just mentioned.

**EXAMPLE 1.** Let $T$ be a self-mapping on $[0,1]$ defined by

$$T(0) = 1, \ T(1) = 0, \ T(x) = \frac{1}{2}, \ x \epsilon (0,1).$$

**EXAMPLE 2.** Let $T$ be a self-mapping on $[0,1]$ defined by $T(x) = 1-x, \ x \epsilon [0,1]$.

**EXAMPLE 3.** Let $T$ be a self-mapping on $[-1,1]$ defined by $T(x) = -x, \ x \epsilon [-1,1]$.

The mapping $T$ of the above examples satisfies (1.1) for $\lambda = \frac{1}{2}$. However, for $x=0$, $y=1$, $T$ of Example 1 or Example 2, and for $x=1$, $y=-1$, $T$ of Example 3 do not satisfy the above well-known contractions. Next, we define generalized mean value non-expansive mapping: Let $T$ be a self-mapping of a Banach space $E$. Then $T$ will be called a generalized mean value non-expansive mapping if for any $x, y$ in $E$, there exists non-negative real numbers $a_i$ $(i = 1, 2, \ldots, 5)$ such that

$$|TT_\lambda^k x - TT_\lambda^k y| \leq a_1|x-y| + a_2|TT_\lambda^k x| + a_3|y - TT_\lambda^k y| + a_4|TT_\lambda^k y| + a_5|y - TT_\lambda^k x|,$$

(1.2)

where $\sum_{i=1}^{5} a_i = 1$ and $TT_\lambda^k = \lambda x + (1-\lambda) TT_\lambda x, \ 0 < \lambda \leq 1$ holds.

Now we define a new contraction which is more general than (1.1) as follows:

Let $X$ be subset of a normed linear space $E$. A mapping $T: X \rightarrow X$ is called an iteratively mean value contraction mapping if for every $x \in X$ there exist non-negative real numbers $a$, such that

$$|TT_\lambda (TT_\lambda x) - TT_\lambda x| \leq a|TT_\lambda x - x|,$$

(1.3)

where $0 < \lambda \leq 1$ and $TT_\lambda x = \lambda x + (1-\lambda) T_\lambda x$ and $TT_\lambda x = T (\lambda x + (1-\lambda) T x)$ holds.

The above definition is given because there are self-mappings of a subset of a normed linear space, which do not satisfy (1.1), but satisfies (1.3). An example of self-mapping for which (1.3) holds but (1.1) does not hold, is given below:

**EXAMPLE 4.** Let $T$ be a self-mapping on $[-1,7]$ defined by

$$Tx = -x, \ x \in [-1,1], \ Tx = \frac{6}{7} - x, \ x \in [1,7].$$
2. **MAIN THEOREMS.**

**THEOREM 1.** Let $T$ be a self-mapping of a normed linear space $E$. If

(i) $T$ satisfies (1.1),

(ii) $\{x_n\}$ converges to $u \in E$ where $x_n = TT_\lambda x_{n-1}$ $n=1,2,\ldots$ for any $x_0 \in E$,

(iii) $T(\lambda u + (1-\lambda) Tu) = \lambda Tu + (1-\lambda) T^2 u$, only for $u$;

then $T$ has a unique fixed point in $E$.

**PROOF:** Let $x_0$ be any point in $E$. Define, $x_n = TT_\lambda x_{n-1}$ $(n = 1,2,\ldots)$. Put

$x_0 = x$ and $x_1 = y$ in (1.1), then we have

$$||x_1 - x_2|| \leq a_1 ||x_0 - x_1|| + a_2 ||x_0 - x_1|| + a_3 ||x_1 - x_2|| + a_4 ||x_0 - x_2|| , \quad (2.1)$$

Again, put $x_1 = x$ and $y = x_0$ in (1.1). Then

$$||x_2 - x_1|| \leq a_1 ||x_1 - x_0|| + a_2 ||x_1 - x_2|| + a_3 ||x_0 - x_1|| + a_5 ||x_0 - x_2|| . \quad (2.2)$$

Adding (2.1) and (2.2), we obtain

$$||x_2 - x_1|| \leq r ||x_1 - x_0|| ,$$

where $r = \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5}$ and $r < 1$, since $\sum_{i=1}^{5} a_i < 1$.

By induction it may be proved that

$$||x_n - x_{n+1}|| \leq r^n ||x_1 - x_0||$$

It may be shown by routine calculation that $\{x_n\}$ is a Cauchy sequence. Hence $\{x_n\}$ is convergent. So, by (ii), $x_n \to u \in E$, as $n \to \infty$.

Now,

$$||u - TT_\lambda u|| \leq ||u - x_{n+1}|| + ||TT_\lambda x_n - TT_\lambda u||$$

$$\leq ||u - x_{n+1}|| + a_1 ||x_n - u|| + a_2 ||x_n - x_{n+1}|| + a_3 ||u - TT_\lambda u|| + a_4 ||x - TT_\lambda u|| + a_5 ||u - x_{n+1}||$$

$$\leq (a_1 + a_2 + a_3 + a_4) ||u - TT_\lambda u|| , \quad \text{as} \quad n \to \infty .$$

Therefore, $(1 - a_1 - a_2 - a_3 - a_4) ||u - TT_\lambda u|| \leq 0$, which implies that $u = TT_\lambda u$, since $\sum_{i=1}^{5} a_i < 1$. Now, $Tu = T(TT_\lambda u) = T(T(\lambda u + (1-\lambda) Tu)) = T(\lambda Tu + (1-\lambda) T^2 u)$, by (1).

Therefore,

$$||u - Tu|| = ||T(\lambda u + (1-\lambda) Tu) - T(\lambda Tu + (1-\lambda) T^2 u)|| \leq r ||u - Tu||,$$

by (i).

Since $r < 1$, $(1 - r) ||u - Tu|| \leq 0$ implies $Tu = u$ i.e. $u$ is a fixed point of $T$.

Uniqueness of the fixed point follows easily.
THEOREM 2. Let $T$ be a self-mapping of a bounded convex subset $M$ of a normed linear space $E$. If for any $x \in M$,

(i) $T$ satisfies (1.3)

(ii) \{${x_n}$\} converges to $u \in M$, whenever \{${x_n}$\} is convergent, where $x_n = TT^\lambda x_{n-1}$, 
\(n = 1,2,3,\ldots\) for any $x_0 \in M$.

(iii) $\lim_{n \to \infty} T(\lambda x_n + (1-\lambda) T x_n) = T(\lambda \lim_{n \to \infty} x_n + (1-\lambda) T \lim_{n \to \infty} x_n)$

(iv) $T(\lambda u + (1-\lambda) Tu) = \lambda Tu + (1-\lambda) T^2 u$, for all $u$;

then $T$ has a fixed point.

PROOF: Proof is exactly similar to that of Theorem 1, so we omit it.

THEOREM 3. Let $E$ be a rotund Banach space, $M$ be a compact convex subset of $E$ and $T$ be a self-mapping of $M$. If $T$ is continuous and $T$ satisfies (1.2) and $TT^\lambda x = T^\lambda Tx$ for any $x \in M$, then $T$ has a fixed point in $M$.

PROOF: Let $x$ be any point in $M$. Define $f(x) = \|x - Tx\|$. Since $T$ and $\|\cdot\|$ are continuous functions, therefore, $f(x)$ is also continuous. So $f(x)$ attains its minimum for some $x$ (say $x = z \in M$).

First suppose $\|Tz - z\| = 0$, then $z$ is a fixed point of $T$. Now let $\|Tz - z\| \neq 0$. Hence

$$f(T^\lambda z) = ||T^\lambda z - T(T^\lambda z)|| = ||T^\lambda z - TT^\lambda(Tz)||$$

$$\leq ||z - Tz|| < ||z - Tz||, \text{ since } E \text{ is rotund.}$$

$= f(z)$, which contradicts the minimality of $f(z)$.

Therefore $\|T(z) - z\| = 0$ i.e. $Tz = z$ is a fixed point of $T$.

THEOREM 4. Let $E$ be a Banach space, $M$ be a compact convex subset of $E$, and $T$ be a continuous self-mapping of $M$. If for any $x,y (x \neq y) \in M$, $T$ satisfies (1.1) (where $\leq$ is replaced by $<$) and $\sum_{i=1}^{5} a_i = 1$ and $TT^\lambda x = T^\lambda Tx$, then $T$ has a unique fixed point in $M$.

PROOF: Proof is similar to that of Theorem 3.

3. CONCLUDING REMARKS.

(i) That the condition (iii) of Theorem 1 is necessary for existence of fixed point of $T$ as illustrated by the following example.
EXAMPLE 4. Let $T$ be a self-mapping on $[0,1]$ defined by $T x = 1 - x$, $x \in [0,1]$, $T(1) = 0$. Here $T$ satisfies conditions (i) and (ii) of Theorem 1 for $\lambda < 1$, but it does not satisfy (iii) and $T$ has no fixed point in $[0,1]$.

(ii) The self-mapping $T$ of Example 1 and Example 2 are non-expansive ($||T x - T y|| \leq ||x - y||$). Kirk [4] has proved the following fixed point theorem on non-expansive mapping:

"If $K$ be a nonempty closed convex bounded subset of a reflexive Banach space $X$ and if $K$ posseses normal structure, then every non-expansive mapping from $K$ into itself has a fixed point."

The same result is also established independently by Browder [5] in a uniformly convex Banach space. There is a close connection between the theorems of Kirk and Browder. This was first noted by Goebel [6] that if $X$ be a uniformly convex Banach space, then any closed convex bounded subset $K$ of $X$, must have normal structure.

We observe that for the existence of a fixed point of any non-expansive mapping in a Banach space, the Banach space must have a property either "uniform convexity" or "reflexivity with normal structure". Though self-mapping $T$ in Example 1 and Example 2 are non-expansive, they are contractions in the sense (1.1). These mappings satisfy all the conditions of Theorem 1. Theorem 1 explains the existence of the fixed point of the above mappings without assuming "uniform convexity" or "reflexivity with normal structure".

These examples also suggest that non-expansive mappings may be converted into contraction mappings (general process of conversion is not known). Since the study of contraction mappings is easier than non-expansive mapping, so this type conversion has some importance in fixed point theory.

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