K-SPACE FUNCTION SPACES

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ABSTRACT. A study is made of the properties on X which characterize when \( C(X) \) is a k-space, where \( C(X) \) is the space of real-valued continuous functions on X having the topology of pointwise convergence. Other properties related to the k-space property are also considered.

KEY WORDS AND PHRASES. Function spaces, k-spaces, Sequential spaces, Fréchet spaces, Countable tightness, k-countable, \( \tau \)-countable.

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1. INTRODUCTION.

If \( X \) is a topological space, the notation \( C(X) \) is used for the space of all real-valued continuous functions on \( X \). One of the natural topologies on \( C(X) \) is the topology of pointwise convergence, where subbasic open sets are those of the form

\[
\bigcap x, V = \{f \in C(X) | f(x) \in V\}
\]
for \( x \in X \) and \( V \) open in the space of real numbers, \( \mathbb{R} \), with the usual topology. The space \( C(X) \) with the topology of pointwise convergence will be denoted by \( C(X) \).

For a completely regular space \( X \), \( C(X) \) is first countable, in fact metrizable, if and only if \( X \) is countable [2]. The purpose of this paper is to show to what extent this result can be extended to properties more general than first countability, such as that of being a \( k \)-space. Throughout this paper all spaces will be assumed to be completely regular \( T_1 \)-spaces.

We first recall the definitions of certain generalizations of first countability. The space \( X \) is a Frechet space if whenever \( x \in \overline{A} \subseteq X \), there exists a sequence in \( A \) which converges to \( x \). The space \( X \) is a sequential space if the open subsets of \( X \) are precisely those subsets \( U \) such that whenever a sequence converges to an element of \( U \), the sequence is eventually in \( U \). Also \( X \) is a \( k \)-space if the closed subsets of \( X \) are precisely those subsets \( A \) such that for every compact subspace \( K \subseteq X \), \( A \cap K \) is closed in \( K \). Finally \( X \) has countable tightness if whenever \( x \in \overline{A} \subseteq X \), there exists a countable subset \( B \subseteq A \) such that \( x \in \overline{B} \). The following diagram shows the implications between these properties.

\[
\begin{array}{ccc}
\text{first countable} & \downarrow & \\
\text{Frechet} & \longrightarrow & \text{sequential} \quad \longrightarrow \quad \text{k-space} \quad \downarrow & \text{countable tightness} \\
\end{array}
\]

We will show that the Frechet space, sequential space, and \( k \)-space properties are equivalent for \( C(X) \). In order to characterize these properties for \( C(X) \) in terms of internal properties of \( X \), we will need to make some additional definitions. Let \( \mathcal{S}(X) \) be the set of all nonempty finite subsets of \( X \). A collection \( \mathcal{U} \) of open subsets of \( X \) is an open cover for finite subsets of \( X \) if for every \( A \in \mathcal{S}(X) \), there exists a \( U \in \mathcal{U} \) such that \( A \subseteq U \). If \( \{U_n\} \) is a sequence of collections of subsets of \( X \), a string from \( \{U_n\} \) is a sequence \( \{U_n\} \) such that \( U_n \in U_n \).
for every $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers). In addition, we will say that $\{U_n\}$ is **residually covering** if for every $x \in X$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x \notin U_n$.

**THEOREM 1.** The following are equivalent.

(a) $C_\pi(X)$ is a Fréchet space.
(b) $C_\pi(X)$ is a sequential space.
(c) $C_\pi(X)$ is a k-space.
(d) Every sequence of open covers for finite subsets of $X$ has a residually covering string.

**PROOF.** (d) $\Rightarrow$ (a). Suppose that every sequence of open covers for finite subsets of $X$ has a residually covering string. Let $F$ be a subset of $C_\pi(X)$, and let $f$ be an accumulation point of $F$ in $C_\pi(X)$. Then for every $n \in \mathbb{N}$ and $A = \{x_1, \ldots, x_k\} \in \mathcal{P}(X)$, we may choose an $f_{n,A} \in F \cap \bigcap_{i=1}^{k} \left( (f(x_i) - \frac{1}{n}, f(x_i) + \frac{1}{n}) \right) \cap \left( (f(x_k) - \frac{1}{n}, f(x_k) + \frac{1}{n}) \right)$.

Also define $U(n, A) = \{x \in X| |f_{n,A}(x) - f(x)| < \frac{1}{n}\}$, which is an open subset of $X$. Then for each $n \in \mathbb{N}$, define $u_n = \{U(n, A) | A \in \mathcal{P}(X)\}$, which is an open cover for finite subsets of $X$. Now $\{u_n\}$ has a residually covering string $\{U(n, A_n)\}$, so that for every $n \in \mathbb{N}$, we may define $f_n = f_{n,A_n}$.

We wish to establish that $\{f_n\}$ converges to $f$ in $C_\pi(X)$. So let $x \in X$, and let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ with $N \geq \frac{1}{\varepsilon}$ such that for every $n \geq N$, $x \in U(n, A_n)$. But then if $n > N$,

$$|f_n(x) - f(x)| = |f_{n,A_n}(x) - f(x)| < \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon.$$  

Therefore $\{f_n(x)\}$ converges to $f(x)$ for every $x \in X$, so that $\{f_n\}$ converges to $f$ in $C_\pi(X)$. Hence $C_\pi(X)$ must be a Fréchet space.

(c) $\Rightarrow$ (d). Suppose $X$ has a sequence $\{U_n\}$ of open covers for finite subsets such that no string from $\{U_n\}$ is residually covering. Let $V_1 = U_1$, and for each $n > 1$, let $V_n$ be an open cover for finite subsets of $X$ which refines both $V_{n-1}$
and $U_n$. For every $n \in \mathbb{N}$ and $A \in \mathcal{F}(X)$, let $U(n,A) \in V_n$ such that $A \subseteq U(n,A)$, and let $f_{n,A} \in C(X)$ be such that $f_{n,A}(A) = \{\frac{1}{n}\}$, $f_{n,A}(X \setminus U(n,A)) = \{\frac{1}{n}\}$, and $f_{n,A}(x) \subseteq \left[\frac{1}{n}, \frac{1}{n+1}\right]$. Then define

$$F = \{f_{n,A} | n \in \mathbb{N} \text{ and } A \in \mathcal{F}(X)\},$$

and also define $F^* = \overline{F \setminus \{c_0\}}$ in $C_\pi(X)$, where $c_0$ is the constant zero function.

First we establish that $F^*$ is not closed in $C_\pi(X)$ by showing that $c_0$ is an accumulation point of $F$ in $C_\pi(X)$. To do this, let $W = \mathbb{I}_{x_1, V_1} \cap \ldots \cap \mathbb{I}_{x_k, V_k}$ be an arbitrary basic neighborhood of $c_0$ in $C_\pi(X)$. If $A = \{x_1, \ldots, x_k\}$ and $n \in \mathbb{N}$ such that $\frac{1}{n} \in V_1 \cap \ldots \cap V_k$, then $f_{n,A} \in W \cap F$.

We will then obtain that $C_\pi(X)$ is not a k-space, as desired, if we can show that the intersection of $F^*$ with each compact subspace of $C_\pi(X)$ is closed in that compact subspace. To this end, let $K$ be an arbitrary compact subspace of $C_\pi(X)$. Then for every $x \in X$, the orbit $\{f(x) | f \in K\}$ is bounded in $\mathbb{R}$. For every $x \in X$, define $M(x) = \sup \{f(x) | f \in K\}$, and also for every $m \in \mathbb{N}$, define $X_m = \{x \in X | M(x) \leq m\}$. Note that $X = \bigcup \{X_m | m \in \mathbb{N}\}$, and that for every $m$, $X_m \subseteq X_{m+1}$.

Suppose, by way of contradiction, that for every $m, n \in \mathbb{N}$, there exists a $k \geq n$ and $V \in V_k$ such that $X_m \subseteq V$. We define, by induction, a string $\{U_n\}$ from $\{U_n\}$. First there exists a $k_1 \geq 1$ and $V_1 \in V_{k_1}$ such that $X_1 \subseteq V_1$. For each $i = 1, \ldots, k_1$, choose $U_i \in U_1$ so that $V_1 \subseteq U_i$. Now suppose $k_m$ and $U_1, \ldots, U_{k_m}$ have been defined. Then there exists a $k_{m+1} \geq k_m + 1$ and $V_{m+1} \in V_{k_{m+1}}$ such that $X_{m+1} \subseteq V_{m+1}$. For each $i = k_m + 1, \ldots, k_{m+1}$, choose $U_i \in U_i$ so that $V_{m+1} \subseteq U_i$. This defines string $\{U_n\}$, which we know to not be residually covering. Let $x \in X$ be arbitrary. There is an $m \in \mathbb{N}$ such that $x \in X_m$. Let $n \geq k_m$. There is a $j \geq m$ such that $k_{j-1} + 1 \leq n \leq k_j$. Then $x \in X_m \subseteq X_j \subseteq V_j \subseteq U_n$. But this says that $\{U_n\}$ is residually covering, which is a contradiction.

We have just established that there exist $m, n \in \mathbb{N}$ such that for every $k \geq n$ and for every $V \in V_k$, $X_m \notin V$. Then define $M = \max \{m, n\}$, let $x_0 \in X$ be
arbitrary, and define $W = \prod_{x_0, \left(\frac{-1}{M}, \frac{1}{M}\right)}$, which is a neighborhood of $c_0$ in $C(X)$. Suppose $f \in W \cap F$. Then there exists a $k \in \mathbb{N}$ and $A \in \mathcal{F}(X)$ such that $f = f_{k,A}$. Since $\frac{1}{k} \leq f(x_0) < \frac{1}{M}$, then $k > M \geq n$. Thus $x_m \notin U(k,A)$, so that there exists an $x_1 \in X \setminus U(k,A)$. But then $f(x_1) = k > M \geq n \geq M(x_1)$, so that $f \notin K$. Therefore $W \cap F \cap K = \emptyset$, so that $c_0$ is not an accumulation point of $F \cap K$ in $K$. Hence $F \cap K$ must be closed in $K$. Since $K$ was arbitrary, we obtain that $C_\pi(X)$ is not a $k$-space.

**THEOREM 2.** $C_\pi(X)$ has countable tightness if and only if every open cover for finite subsets of $X$ has a countable subcover for finite subsets of $X$.

**PROOF.** Suppose that every open cover for finite subsets of $X$ has a countable subcover for finite subsets of $X$. Let $F$ be a subset of $C_\pi(X)$, and let $f$ be an accumulation point of $F$ in $C_\pi(X)$. Then for each $n \in \mathbb{N}$ and $A = \{x_1, \ldots, x_k\} \in \mathcal{F}(X)$, choose

$$f_{n,A} \in F \cap \prod_{x_1, (f(x_1) - \frac{1}{n}, f(x_1) + \frac{1}{n})} \cap \prod_{x_k, (f(x_k) - \frac{1}{n}, f(x_k) + \frac{1}{n})}.$$ 

Also let $U(n,A) = \{x \in X \mid |f_{n,A}(x) - f(x)| < \frac{1}{n}\}$, which is an open subset of $X$. Then for each $n \in \mathbb{N}$, $\{U(n,A) \mid A \in \mathcal{F}(X)\}$ is an open cover for finite subsets of $X$. So for each $n \in \mathbb{N}$, there exists a sequence $\{A(n,i) \mid i \in \mathbb{N}\}$ from $\mathcal{F}(X)$ such that $\{U(n,A(n,i)) \mid i \in \mathbb{N}\}$ is a cover for finite subsets of $X$. Then define $G = \{f_{n,A(n,i)} \mid n,i \in \mathbb{N}\}$.

To see that $f \in G$, let $W = \prod_{x_1, V_1} \cap \cdots \cap \prod_{x_k, V_k}$ be a neighborhood of $f$ in $C_\pi(X)$. Let $A = \{x_1, \ldots, x_k\}$, and choose $n \in \mathbb{N}$ so that $(f(x_j) - \frac{1}{n}, f(x_j) + \frac{1}{n}) \subseteq V_j$ for each $j = i, \ldots, k$. Then there is an $i \in \mathbb{N}$ such that $A \subseteq U(n,A(n,i))$. So for each $x \in A$, $|f_{n,A(n,i)}(x) - f(x)| < \frac{1}{n}$, and hence $f_{n,A(n,i)} \in W$.

Conversely, suppose that $C_\pi(X)$ has countable tightness, and let $U$ be an open cover for finite subsets of $X$. For each $A \in \mathcal{F}(X)$, let $U(A) \in U$ be such that $A \subseteq U(A)$. Also for each $n \in \mathbb{N}$ and $A \in \mathcal{F}(X)$, let $f_{n,A} \in C(X)$ be such that
Then define \( F = \{ f_{n,A} \mid n \in \mathbb{N} \text{ and } A \in \tau(X) \} \).

Since the constant zero function, \( c_0 \), is an accumulation point of \( F \), then there is a countable subset \( G \) of \( F \) such that \( c_0 \in G \). There are sequences \( \{ n_i \} \subseteq \mathbb{N} \) and \( \{ A_{i} \} \subseteq \tau(X) \) so that \( G = \{ f_{n_i,A_i} \mid i \in \mathbb{N} \} \).

To see that \( \{ U(A_i) \mid i \in \mathbb{N} \} \) is a cover for finite subsets of \( X \), let \( A = [x_1, \ldots, x_k] \in \tau(X) \). Then there exists an \( i \in \mathbb{N} \) such that \( f_{n_i,A_i} \subseteq [x_1, \ldots, x_k] \subseteq \tau(X) \). But this means that \( A \subseteq U(A_i) \), so that \( \{ U(A_i) \mid i \in \mathbb{N} \} \) is indeed a cover for finite subsets of \( X \).

Let us now give names to the two properties of \( X \) which are expressed in Theorems 1 and 2. We will call \( X \) \( k \text{-countable} \) whenever \( \tau(X) \) is a \( k \)-space, and we will call \( X \) \( \tau \text{-countable} \) whenever \( \tau(X) \) has countable tightness. We state some immediate facts about these properties.

**Proposition 3.** Every countable space is \( k \)-countable.

**Proposition 4.** Every \( k \)-countable space is \( \tau \)-countable.

**Proposition 5.** Every \( \tau \)-countable space is Lindelöf.

**Proof.** Let \( X \) be \( \tau \)-countable, and let \( U \) be an open cover of \( X \). Let \( V \) be the family of all finite unions of members of \( U \). Then \( V \) is an open cover for finite subsets of \( X \), so that it has a countable subcover \( \mathcal{H} \) for finite subsets of \( X \). Each member of \( \mathcal{H} \) is a finite union of members of \( U \), so that since \( \mathcal{H} \) covers \( X \), then \( U \) has a countable subcover. \( \Box \)

This means that if \( \tau(X) \) has countable tightness, \( X \) must be Lindelöf. In particular, \( \tau(\omega_0) \) does not have countable tightness, where \( \omega_0 \) is the space of countable ordinals with the order topology. This is in contrast to \( \tau(\omega) \), which we see from the next proposition has countable tightness, where \( \omega = \omega_0 \cup \{ \omega_1 \} \).

**Proposition 6.** If \( X^n \) is Lindelöf for every \( n \in \mathbb{N} \), then \( X \) is \( \tau \)-countable.

**Proof.** Let \( X^n \) be Lindelöf for every \( n \in \mathbb{N} \), and let \( U \) be an open cover for finite subsets of \( X \). For each \( n \in \mathbb{N} \), let \( U_n = \{ U^n \subseteq X^n \mid U \in U \} \). Since \( U \) is an
open cover for finite subsets of $X$, then each $U_n$ is an open cover of $X^n$. So for each $n \in \mathbb{N}$, $U$ has a countable subcollection $V_n$ such that \{$U^n| U \in V_n\}$ covers $X^n$. But then $\bigcup \{V_n|n \in \mathbb{N}\}$ is a countable subcollection of $U$ which is a cover for finite subsets of $X$. □

**COROLLARY 7.** Every compact space is $\tau$-countable, and every separable metric space is $\tau$-countable.

We now examine some properties of $k$-countable spaces.

**PROPOSITION 8.** Every closed subspace of a $k$-countable space is $k$-countable.

**PROOF.** Let $X$ be a $k$-countable space, and let $Y$ be a closed subspace of $X$.

Let $\{V_n\}$ be a sequence of open covers for finite subsets of $Y$. For each $n \in \mathbb{N}$, let $U_n = \{V \cup (X \setminus Y)|V \in V_n\}$, which is an open cover for finite subsets of $X$. Now $\{U_n\}$ has a residually covering string $\{V_n \cup (X \setminus Y)\}$, where each $V_n \in V_n$. But then $\{V_n\}$ is a residually covering string from $\{V_n\}$. □

**PROPOSITION 9.** Every continuous image of a $k$-countable space is $k$-countable.

**PROOF.** Let $X$ be $k$-countable, and let $f : X \to Y$ be a continuous surjection.

Let $\{V_n\}$ be a sequence of open covers for finite subsets of $Y$. For each $n \in \mathbb{N}$, let $U_n = \{f^{-1}(V)|V \in V_n\}$, which is an open cover for finite subsets of $X$. Now $\{U_n\}$ has a residually covering string $\{f^{-1}(V_n)\}$, where each $V_n \in V_n$. But then $\{V_n\}$ is a residually covering string from $\{V_n\}$. □

In the next proposition, we use the term covering string, by which we mean a string which is itself a cover of the space.

**PROPOSITION 10.** If $X$ is $k$-countable, then every sequence of open covers of $X$ has a covering string.

**PROOF.** Let $\{U_n\}$ be a sequence of open covers of $X$. For each $n \in \mathbb{N}$, let $V_n = \{U_n \cup \ldots \cup U_{n+k+1}|k \in \mathbb{N}$ and each $U_i \in U_i\}$, which is an open cover for finite subsets of $X$. Thus $\{V_n\}$ has a residually covering string $\{V_n\}$. Now $V_1 = U_1 \cup \ldots \cup U_{k_1}$ for some $k_1 \in \mathbb{N}$. Also $V_{k_1+1} = \ldots$
Continuing by induction, we can define an increasing sequence \( \{k_i\} \) such that each \( V_{k_{i+1}} = U_{k_{i+1}} \cup \cdots \cup U_{k_{i+1}} \). This defines \( U_n \) for each \( n \in \mathbb{N} \). To see that \( \{U_n\} \) is a covering string from \( \{U_n\} \) let \( x \in X \). Then there exists an \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( x \in V_n \). Since \( \{k_i\} \) is increasing, there is some \( i \) such that \( k_i \geq N \). Then \( x \in V_{k_{i+1}} = U_{k_{i+1}} \cup \cdots \cup U_{k_{i+1}} \), so that \( x \) is indeed in some \( U_n \).

We next give an important example of a space which is not \( k \)-countable.

**Example 11.** The closed unit interval, \( I \), is not \( k \)-countable.

**Proof.** For each \( n \in \mathbb{N} \), let \( U_n \) be the set of all open intervals in \( I \) having diameter less than \( \frac{1}{2^n} \). Suppose \( \{U_n\} \) were to have a covering string \( \{U_n\} \). Then since \( I \) is connected, there would be a simple chain \( \{U_{n_1}, \ldots, U_{n_k}\} \) from 0 to 1. That is, \( 0 \in U_{n_1}, 1 \in U_{n_k} \), and for each \( 1 \leq i \leq k-1 \), there is a \( t_i \in U_{n_i} \cap U_{n_{i+1}} \). But then

\[
1 \leq |t_1 - t_{k-1}| + |t_{k-1} - t_{k-2}| + \ldots + |t_2 - t_1| + |t_1|
\]

\[
< \frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \ldots + \frac{1}{2^2} + \frac{1}{2^1}
\]

\[
< \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k} < 1.
\]

This is a contradiction, so that \( \{U_n\} \) cannot have a covering string. Therefore, by Proposition 10, \( I \) is not \( k \)-countable. \(\square\)

The next three results are consequences of Example 11.

**Example 12.** The Cantor set, \( K \), is not \( k \)-countable.

**Proof.** Since there exists a continuous function from \( K \) onto \( I \), then \( K \) cannot be \( k \)-countable because of Proposition 9 and Example 11. \(\square\)

Our next proposition then follows from Example 12 and Proposition 8.

**Proposition 13.** No \( k \)-countable space contains a Cantor set.

**Proposition 14.** Every \( k \)-countable space is \( \sigma \)-dimensional.
PROOF. Let $X$ be $k$-countable, let $x \in X$, and let $U$ be an open neighborhood of $x$ in $X$. Since $X$ is completely regular, there exists an $f \in C(X)$ such that $f(x) = 0$, $f(X \setminus U) = \{0\}$, and $f(X) \subseteq \mathbb{I}$. Since $\mathbb{I}$ is not $k$-countable by Example 11, and since $f(X)$ is $k$-countable by Proposition 9, then there exists a $t \in \mathbb{T} \setminus f(X)$. Thus $[0,t) \cap f(X)$ is both open and closed in $f(X)$, so that $f^{-1}([0,t))$ is an open and closed neighborhood of $x$ contained in $U$. 

With all these necessary conditions which $k$-countable spaces must satisfy, one might wonder whether there exists an uncountable $k$-countable space. This is answered by the next two examples.

We will call a space $X$ virtually countable if there exists a finite subset $F$ of $X$ such that for every open subset $U$ of $X$ with $F \subseteq U$, it is true that $X \setminus U$ is countable. Notice that a first countable virtually countable space is countable.

**PROPOSITION 15.** Every virtually countable space is $k$-countable.

**PROOF.** Let $F$ be a finite subset of $X$ such that every open $U$ in $X$ with $F \subseteq U$ has countable complement, and let $\{U_n\}$ be a sequence of open covers for finite subsets of $X$. First let $U_1 \subseteq U_1$ be such that $F \subseteq U_1$. Then $X \setminus U_1$ is countable; say $X \setminus U_1 = \{x_{11}, x_{12}, x_{13}, \ldots\}$. Let $U_2 \subseteq U_2$ be such that $F \cup \{x_{11}\} \subseteq U_2$. Now $X \setminus U_2$ is also countable; say $X \setminus U_2 = \{x_{21}, x_{22}, x_{23}, \ldots\}$. Let $U_3 \subseteq U_3$ be such that $F \cup \{x_{11}, x_{12}, x_{21}\} \subseteq U_3$. Continuing by induction, we may define string $\{U_n\}$ from $\{U_n\}$ such that for each $n$, $U_n = X \setminus \{x_{n1}, x_{n2}, x_{n3}, \ldots\}$ and

$$F \cup \{x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2,n-1}, \ldots, x_{n1}\} \subseteq U_{n+1}.$$  

To see that every element of $X$ is residually in $\{U_n\}$, let $x \in X$. If $x \in \bigcap_{n=1}^{\infty} U_n$, then $x$ is residually in $\{U_n\}$. If $x \notin \bigcap_{n=1}^{\infty} U_n$, then let $i$ be the first integer such that $x \notin U_i$. Then $x = x_{ij}$ for some $j$, so that for every $n \geq i + j$, $x \in U_n$. Therefore $x$ is residually in $\{U_n\}$. 

EXAMPLE 16. The space of ordinals, $\Omega$, which are less than or equal to the first uncountable ordinal is $k$-countable.

PROOF. It is easy to see that $\Omega$ is virtually countable. □

EXAMPLE 17. The Fortissimo space, $\mathbb{F}$, is $k$-countable, where $\mathbb{F}$ is $\mathbb{R}$ with the following topology: each $\{t\}$ is open for $t \neq 0$, and the open sets containing 0 are the sets containing 0 which have countable complements. Also $\mathbb{F}^2$ is not Lindelöf, which shows that the converse of Proposition 6 is not true.

PROOF. Obviously $\mathbb{F}$ is virtually countable. However, an alternate proof can be obtained from known properties of this space. In particular, it follows from [1] that $C_\pi(\mathbb{F})$ is homeomorphic to a $\Sigma$-product of copies of $\mathbb{R}$, and from [3] that a $\Sigma$-product of first countable spaces is a Fréchet space. □

The spaces in the previous two examples are not first countable. This raises the following question.

QUESTION 18. Is every first countable $k$-countable space countable?

One well studied example of an uncountable first countable space which is also a $\sigma$-dimensional Lindelöf space and which does not contain a Cantor set is the Sorgenfrey line. However, in our last example we show that this space is not $k$-countable, and in fact is not even $\tau$-countable.

EXAMPLE 19. The Sorgenfrey line, $S$, is not $\tau$-countable. This shows that the converse of Proposition 5 is not true.

PROOF. For each $A \in \mathcal{P}(S)$, let $\delta(A) = \frac{1}{2} \min \{ |a-a'| | a, a' \in A, \text{ with } a \neq a' \}$, and let $U(A) = \bigcup \{ [a, a + \delta(A)] | a \in A \}$. Then define $\mathcal{U} = \{ U(A) | A \in \mathcal{P}(S) \}$, where $A \cup \{ -a | a \in A \}$. Clearly $\mathcal{U}$ is an open cover for finite subsets of $S$. Then $\{ U^2 | U \in \mathcal{U} \}$ is an open cover of $S^2$. But each $U^2$, for $U \in \mathcal{U}$, intersects the set $\{ (x, y) \in S^2 | x + y = 0 \}$ on a finite set, so that $\{ U^2 | U \in \mathcal{U} \}$ has no countable subcover of $S^2$. Therefore no countable subcollection of $\mathcal{U}$ can cover all doubleton subsets of $S$. □
REFERENCES

