A LEBSGUE DECOMPOSITION FOR ELEMENTS IN
A TOPOLOGICAL GROUP

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ABSTRACT. Our aim is to establish the Lebesgue decomposition for strongly-bounded elements in a topological group. In 1963 Richard Darst established a result giving the Lebesgue decomposition of strongly-bounded elements in a normed Abelian group with respect to an algebra of projection operators. Consequently, one can establish the decomposition of strongly-bounded additive functions defined on an algebra of sets. Analogous results follow for lattices of sets. Generalizing some of the techniques yield decompositions for elements in a topological group.

KEY WORDS AND PHRASES. Lebesgue decomposition, projection operator, strongly-bounded, topological group.

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1. INTRODUCTION.

In 1963 R. B. Darst [2] established a result giving the Lebesgue decomposition
of s-bounded elements in a normed Abelian group with respect to an algebra of projection operators. Consequently, one can establish the decomposition of s-bounded additive functions defined on an algebra of sets \([4]\). The set of corresponding restrictions of additive set functions defined on a lattice of sets corresponds to a lattice of projection operators \([5]\). The analogous result on lattices is established by using the same techniques \([3]\). More recently, Traynor has obtained decompositions of set functions with values in a topological group \([6], [7]\). The purpose here is to present a Lebesgue decomposition theorem for elements in a topological group by the use of projection operators. It is believed that this result would aid in obtaining decompositions of operators on non-locally convex lattices.

2. PRELIMINARIES

Let \(G\) be an Abelian topological group under addition, and let \(T\) be an algebra of projection operators \([1]\) on \(G\). For \(t_1, t_2 \in T\) define \(t_1 \leq t_2\) to mean \(t_1 \leq t_2\) and define \(t_1 - t_2\) to mean \(t_1 t_2\). This relation induces a partial ordering on \(T\), which in turn has a lattice structure if we set \(t_1 \land t_2 = \sup \{t \in T: t \leq t_1, t \leq t_2\}\) and \(t_1 \lor t_2 = \inf \{t \in T: t_1 \leq t, t_2 \leq t\}\) providing the sup and inf exist. But, we have \(t_1 \lor t_2 = t_1 + t_2 - t_1 t_2 = (t_1 t_2)\) and \(t_1 \land t_2 = t_1 t_2\), so \(T\) is a Boolean algebra of operators. Let \(\mathcal{M}\) be the set of all symmetric neighborhoods about \(0 \in G\). For each \(U \in \mathcal{M}\) and each positive integer \(n\), define \(nU = \{x + y: x, y \in (n-1)U\}\) and \(\mathcal{M}_n = \{0\} \subset G\), whence \(1U = U\). Then a subset \(H \subset G\) is bounded if given \(U \in \mathcal{M}\) there exists an integer \(n\) such that \(H \subset nU\). It would make sense to even say \(H \subset (m/n)U\) for this would mean \(nH \subset mU\). We define an element \(f \in G\) to be s-bounded (strongly bounded) if, for every sequence \(\{t_i\} \subset T\) of pairwise disjoint elements, \(t_i(f) \to 0\). For each positive real number \(x\), \(T_x\) shall denote a non-empty subset of \(T\) with the properties
1) \( t_x \in T_x \) and \( t \in T \) implies \( tt_x \in T_x \), and
2) \( t_x \in T_x \) and \( t_y \in T_y \) implies \( t_x \lor t_y \in T_{x+y} \).

Several lemmas can now be stated, and their proofs follow as in [1] and [2].

**Lemma 1.** Let \( t_1, t_2 \in T \). Suppose \( t_2(g) \in U \) implies \( t_1(g) \in U \) for arbitrary \( g \in G \) and \( U \in \mathcal{U} \). Then \( t_1 \leq t_2 \).

**Lemma 2.** If \( \{t_i\} \) is a monotone sequence of elements of \( T \), and if \( f \in G \) is \( s \)-bounded, then \( \{t_i(f)\} \) is Cauchy in \( G \).

Given \( U \in \mathcal{U} \) we write \( U_0 = U \) and for each \( n > 0 \) we write \( U_n \) to represent some element of \( \mathcal{U} \) where \( U_n + U_n \subseteq U_{n-1} \), whence \( 2^n U_n \subseteq U \). This is possible since addition is continuous in \( G \).

**Definition.** \( T \) has Property A if given \( g \in G \) and \( U \in \mathcal{U} \) then there exists a \( V \in \mathcal{U} \) such that if \( a, b \in T \) and \( (a' b)(g) \notin U \) then \( a(g) \in V \) and \( (a + a' b)(g) \notin V \).

Note that Property A is a condition yielding information about the growth of elements from \( G \); a condition on the manner in which projections affect the relative location of elements in symmetric neighborhoods. We also look at a smaller class of neighborhoods by selecting an arbitrary bounded set \( \hat{U} \) from and forming the sets \( n \hat{U} \) with \( n = 1, 2, \cdots \). Choosing \( \hat{U}_1, \hat{U}_2, \cdots \) we then form \( S = \{\ldots, \hat{U}_2, \hat{U}_1, \hat{U}, 2 \hat{U}, \cdots\} \) and set \( \hat{\mathcal{U}} \) equal to the set

\[
\left\{ \sum_{i=1}^{n} S_i : S_1 \in S, S_i \neq S_j \text{ if } i \neq j \right\}.
\]

It follows that \( \hat{\mathcal{U}} \) possesses the following property inherited from \( \mathcal{U} \): if \( U \in \hat{\mathcal{U}} \) then there exists \( U_1 \in \hat{\mathcal{U}} \) such that \( U_1 + U_1 \subseteq U \). This yields the result,

**Lemma 3.** If \( T \) has Property A with respect to \( \hat{\mathcal{U}} \), and if \( t_1, t_2 \in T \) with \( t_1 \leq t_2 \), then \( t_2(g) \in U \) implies \( t_1(g) \in U \) for arbitrary \( g \in G \) and \( U \in \hat{\mathcal{U}} \).

From now on we shall assume \( T \) has Property A with respect to \( \hat{\mathcal{U}} \).

**Lemma 4.** Let \( f \in G \) be \( s \)-bounded, \( \{t_k\} \subseteq T \) and \( U \in \hat{\mathcal{U}} \). Then there exists a positive integer \( n \) such that if \( j \geq i > n \) then
For \( g \in G \) and \( n \in \mathbb{N} \) let \( S(n, g) = \left\{ U : t(g) \in U \text{ for all } t \in T_{1/n} \right\} \).

Lemma 3 guarantees that no \( S(n, g) \) is empty.

**Lemma 5.** If \( t(g) \neq 0 \) for some \( t \in T_{1/n} \), then there exists a \( W \in S(n, g) \) such that \( W_1 + W_2 + \cdots + W_n \notin S(n, g) \) for all choices of \( W_i \in \hat{\mathcal{U}} \).

**Proof.** Let \( U \in S(n, g) \) and construct a sequence \( \{A_k\} \subset \hat{\mathcal{U}} \) as follows. Set \( A_1 = U_1 + U_2 + \cdots + U_n \) for arbitrary \( U_i \), and set \( A_{k+1} = (A_k)_1 + (A_k)_2 + \cdots + (A_k)_n \) for arbitrary \( (A_k)_i \). Now \( A_1 \subset \left(\frac{(2^n-1)/2^n}{2^n}\right)^{2n} U \), and then \( A_2 \subset \left(\frac{(2^n-1)/2^n}{2^n}\right)^{2n} U \). In general \( A_k \subset \left(\frac{(2^n-1)/2^n}{2^n}\right)^k U \). But the coefficient \( \left(\frac{(2^n-1)/2^n}{2^n}\right)^k \) can be made as small as we like (consequently given \( i > 0 \) there exists \( k > 0 \) such that \( A_k \subset U_i \)), so if the lemma is not true then given \( U \in S(n, g) \) there would exist \( U_1, U_2, \ldots, U_n \) such that \( U_1 + U_2 + \cdots + U_n \notin S(n, g) \). Setting \( A_1 = U_1 + \cdots + U_n \) we apply the hypothesis again and get \( (A_1)_1 + \cdots + (A_1)_n \in S(n, g) \). Continuing this procedure yields sets \( A_k \) which contain \( \{t(g) : t \in T_{1/n}\} \), and which continue to get smaller. This is impossible since \( t(g) \neq 0 \) for some \( t \).

Let us denote this set \( W \) by \( W(n, g) \). This lemma implies that out of all the neighborhoods containing \( \{t(g) : t \in T_{1/n}\} \), \( W(n, g) \) is one of the "smallest."

Since \( \{t(g) : t \in T_{1/(n+1)}\} \) is contained in \( \{t(g) : t \in T_{1/n}\} \) we can choose our \( W(n, g) \) to be nested, \( W(n, g) \supseteq W(n+1, g) \). Assuming this sequence of neighborhoods converges, we are led to defining the following function.

**Definition.** Let \( Y : G \to \hat{\mathcal{U}} \) by \( Y(g) = \lim W(n, g) \).

This function is the counterpart to the function \( y \) in [2]. Our last lemma is the following.

**Lemma 6.** Let \( G \) be complete and \( f \in G \) be \( s \)-bounded. Let \( W(n, f) \) be an associated sequence of neighborhoods as above that contain \( \{t(f) : t \in T_{1/n}\} \). Then given \( M > 0 \) there exists a decreasing sequence \( \{a_i\} \) in \( T \) such that

1) if \( x > 0 \) then there exists an integer \( i \) such that \( a_i \in T_x \), and
2) \( \lim a_i(f) \neq W_1 + W_2 + \cdots + W_M \) where \( W = \lim W(n,f) \), and for all \( W_i \).

**Proof.** To just sketch the essentials of the lemma, we let \( t_i \in T_{1/2^{i+1}} \) such that \( t_i(f) \neq W_1 + W_2 + \cdots + W_{M+2} \) for all \( W_i \), \( i = 1, 2, \ldots, M+2 \). This is possible by the choice of \( W(n,f) \). By Lemma 4 there exists a positive integer \( n_1 \) such that \( j > i > n_1 \) implies \( \bigvee_{i \leq k \leq j} t_k(f) \in W_{M+3} \). Applying Lemma 4 again to the sequence \( t_{n_1+1}, t_{n_1+2}, \ldots, t_{n_2}, \ldots \) produces a positive integer \( n_2 \) such that \( \bigvee_{q \leq k \leq p} t_k(f) \in W_{M+4} \) for \( j > i > n_2 \). Continuing this process we get an increasing sequence \( \{n_j\} \uparrow \) of positive integers such that \( \bigvee_{q \leq k \leq p} t_k(f) \in W_{M+j+2} \) whenever \( p \geq q > n_j \).

If \( u_j = \bigvee_{n_j < i \leq n_{j+1}} t_i \) then \( u_j \in T_{1/2^{n_j+1}} \) and \( k > j \) implies

\[
( \bigvee_{j < p \leq k} u_p - u_j)(f) \in W_{M+j+3}.
\]

Setting \( a_k = \bigwedge_{j \leq k} u_j \) produces the desired decreasing sequence.

We now can state and prove our main decomposition result.

**Theorem.** Let \( G \) be an Abelian topological group, and let \( T \) be an algebra of projection operators on \( G \). Assume \( T_x, \mathcal{M} \) and \( \hat{\mathcal{M}} \) are as before with \( G \) being complete, and with \( T \) possessing Property A with respect to \( \hat{\mathcal{M}} \). If \( f \in G \) is s-bounded then there exists unique elements \( h, s \in G \) such that

1) \( f = h + s \),
2) given \( U \in \hat{\mathcal{M}} \) there exists a positive real number \( x \) such that if \( t \in T_x \) then \( t(h) \in U \),
3) given \( U \in \hat{\mathcal{M}} \) and \( \varepsilon > 0 \) there exists \( t \in T_{\varepsilon} \) such that \( t'(s) \in U \).

**Proof.** First, as counterparts to the classical Lebesgue decomposition theorem, the element \( h \) is to represent the continuous portion of \( f \), while \( s \) represents the singular portion. Again, to just sketch some of the essentials of the proof, we bypass the uniqueness and, turning our attention to existence...
note that if \( h = f \) satisfies condition (2) then there is nothing to prove. Denoting \( Y(f) \) by \( W(f) \), we assume \( W(f) \) contains points other than \( 0 \in G \). Then, from Lemma 6, there exists a sequence \( \{a_{1i}\} \) in \( T \) such that \( \lim a_{1i}(f) \neq W_1(f) + W_2(f) \) for all \( W_1(f) \). Let \( s_1 = \lim a_{1i}(f) \epsilon G \) and \( f_1 = f - s_1 = \lim a_{1i}'(f) \). If \( Y(f_1) = \{0\} \), then \( f_1 \) satisfies (2) and the proof is completed because \( a_{11}(f) = s_1 \) implies \( a_{11}'(s_1) \rightarrow 0 \), and thus \( f_1 \) is also \( s \)-bounded. So given \( U \epsilon \hat{\mathcal{L}} \) and \( \epsilon > 0 \) there exists \( t \epsilon T \) such that \( t'(s) \epsilon U \), namely \( t = a_{1i} \) for large \( i \). If \( f_1 \) does not satisfy (2), applying Lemma 6 to \( f_1 \) produces another sequence \( \{a_{2i}\} \) in \( T \) such that \( \lim a_{2i}(f_1) \neq W_1(f_1) + W_2(f_1) \). Let \( s_2 = \lim a_{2i}(f_1) \) and \( f_2 = f_1 - s_2 = \lim a_{2i}'(f_1) \). Then \( f_2 \) is \( s \)-bounded and \( f = f_2 + (s_1 + s_2) \). To show \( s_1 + s_2 \) satisfies condition (3) we let \( U \epsilon \hat{\mathcal{L}} \) and \( \epsilon > 0 \). We have \( a_{11}'(s_1) \rightarrow 0 \) and \( a_{21}'(s_2) \rightarrow 0 \). So there exists a positive integer \( N \) such that \( a_{11}'(s_1) \epsilon U_1 \) and \( a_{21}'(s_2) \epsilon U_1 \) for all \( i \) greater than \( N \). Then \( (a_{11} \land a_{21}') (s_1 + s_2) = (a_{11}' \land a_{21}') (s_1) + (a_{11}' \land a_{21}') (s_2) \epsilon U \). Condition (3) is satisfied by letting \( t = a_{1i} \lor a_{2i} \) for large \( i \). So if \( Y(f_2) = \{0\} \) then let \( h = f_2 \) and \( s = s_1 + s_2 \) and the proof is completed. If not, continue the process. If for some positive integer \( k \), \( Y(f_k) = \{0\} \), we are through. Otherwise we obtain a sequence \( \{(s_k, f_k)\} \) of pairs of elements of \( G \) and a sequence \( \{\{a_{ki}\} \} \) of non-increasing sequences of elements of \( T \) such that for each positive integer \( k \) we have

1) there exists a sequence \( \{x_{ki}\} \) of positive reals where \( x_{ki} \rightarrow 0 \) and \( a_{ki} \epsilon T \)

2) \( s_k = \lim a_{ki}(f_k - 1) \) with \( f_0 = f \),

3) \( f_k = f_{k-1} - s_k = \lim a_{ki}'(f_{k-1}) \),

4) \( s_k \neq W_1(f_{k-1}) + W_2(f_{k-1}) \) for all \( W_1(f_{k-1}) \),

5) \( f = f_k + \sum_{i=1}^{k} s_i \).
In the end we will have our decomposition \( f = h + s \) with \( s = \sum_{i=1}^{\infty} s_i \) and \( h = f - s \).

Toward this goal, although the steps shall be omitted, the next step is to show \( \lim s_k = 0 \) by showing that \( s_k \) eventually belongs to an arbitrarily selected \( U \in \mathcal{U} \). And then it must be established that \( \lim_{n \to \infty} \sum_{i=1}^{n} s_i \) exists. Assuming this, we then let \( s = \lim_{n \to \infty} \sum_{i=1}^{n} s_i \) and \( h = f - s \). We shall show that \( s \) satisfies condition (3) of the theorem. We have \( s_k = \lim_{i} a_{ki}(f_{k-1}) \). Let \( U \in \mathcal{U} \). Then \( a_{ki}(s_k) \to 0 \), so \( a_{ki}(s_k) \in \bigcup_{k=1}^{\infty} \mathcal{U}_k \) for all \( i \) greater than some positive integer \( M_k \). Since \( s = \lim_{n \to \infty} \sum_{i=1}^{n} s_i \) then there exists a positive integer \( N \) such that \( \sum_{i=1}^{n} s_i \in \mathcal{U}_1 \), and then

\[
\left( \bigvee_{k \leq N} a_{ki} \right)'(s) = \left( \bigvee_{k \leq N} a_{ki} \right)' \sum_{j=1}^{N} s_j + \left( \bigvee_{k \leq N} a_{ki} \right)' \sum_{j > N} s_j
\]

\[
= \sum_{j=1}^{N} k \leq N a_{ki}'(s_j) + \left( \bigvee_{k \leq N} a_{ki} \right)' \sum_{j > N} s_j
\]

\[
\in \mathcal{U}_2 + \cdots + \mathcal{U}_{N+1} + \mathcal{U}_1 \quad \text{for large } i = \max \{ M_1, \ldots, M_N \}
\]

So, let \( t = \bigvee_{k \leq N} a_{ki} \) where \( i = \max \{ M_1, \ldots, M_N \} \) and condition (3) is satisfied. Now \( h = f - s = \lim f_n \) and \( Y(f_n) \to \{ 0 \} \). Then \( Y(h) = \{ 0 \} \) and the decomposition is finished.

These results are part of the author's dissertation from Colorado State University.

REFERENCES


