PEANO COMPACTIFICATIONS AND PROPERTY S METRIC SPACES

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ABSTRACT. Let (X,d) denote a locally connected, connected separable metric space. We say the X is S-metrizable provided there is a topologically equivalent metric ρ on X such that (X,ρ) has Property S, i.e. for any ε > 0, X is the union of finitely many connected sets of ρ-diameter less than ε. It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X, then the usual metric completion (X,ρ) of (X,ρ) is a compact, locally connected, connected metric space, i.e. (X,ρ) is a Peano compactification of (X,ρ). There are easily constructed examples of locally connected connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space (X,d) which has a Peano compactification. In this paper we conjecture that: If (P,ρ) a Peano compactification of (X,ρ|X), X must be S-metrizable. Several (new) necessary and sufficient for a space to be S-metrizable are given, together with an example of non-S-metrizable space which fails to have a Peano compactification.
INTRODUCTION.

Throughout this note let \((X,d)\) denote a locally connected, connected separable metric space. We say that \(X\) is S-metrizable provided there is a topologically equivalent metric \(\rho\) on \(X\) such that \((X,\rho)\) has Property S, i.e. for any \(\varepsilon > 0\), \(X\) is the union of finitely many connected sets of \(\rho\)-diameter less than \(\varepsilon\). It is well-known that S-metrizable spaces are locally connected and that if \(\rho\) is a Property S metric for \(X\), then the usual metric completion \((\tilde{X},\tilde{\rho})\) of \((X,\rho)\) is a compact, locally connected, connected metric space, i.e. \((\tilde{X},\tilde{\rho})\) is a Peano compactification of \((X,\rho)\) [8,p.154].

Property S metric spaces \((X,\rho)\) have been studied extensively in [1,2,3,4,8]. There are easily constructed examples of locally connected, connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space \((X,d)\) which has a Peano compactification. We therefore ask:

QUESTION 1. If \((\rho|X)\) is a Peano compactification of \((X,\rho|X)\), must \(X\) be S-metrizable?

DEFINITIONS AND BASIC RESULTS

A space \(Z\) is an extension of a space \(Y\) if \(Y\) is a dense subspace of \(Z\). If \(Z\) is an extension of \(Y\), we say that \(Y\) is locally connected in \(Z\) if \(Z\) has a basis consisting of regions (that is, open connected sets) whose intersections with \(Y\) are regions in \(Y\). \(Z\) is a perfect extension of \(Y\) if \(Z\) is an extension of \(Y\) and whenever a closed subset \(H\) of \(Y\) separates two sets \(A, B \subseteq Y\) in \(Y\), the set \(\text{cl}_Z H\) (the closure of \(H\) in \(Z\)) separates \(A, B\) in \(Z\). [6]

For completeness we include the following:

THEOREM 2.1 [6]. Let \(Z\) be an extension of \(X\). Then \(X\) is locally connected in \(Z\) if and only if \(Z\) is a perfect locally connected extension of \(X\).

THEOREM 2.2 [6]. Let \((X,d)\) be a metric space. Then \(X\) is S-metrizable if
and only if X has a metrizable compactification Z in which it is locally connected.

**Theorem 2.3** [6]. A topological space is S-metrizable if and only if it has a perfect locally connected metrizable compactification.

**Theorem 2.4** [6]. Let X be a space having a perfect S-metrizable extension. Then X is S-metrizable.

**Theorem 2.5** [5]. Let X be a separable, locally connected, connected rim compact metric space. Then X is S-metrizable.

**Theorem 2.6** [6]. Every countable product of S-metrizable connected spaces $X_1, X_2, \ldots$, is S-metrizable.

### 3. Related Results and Questions.

**Theorem 3.1.** Let $(P, d)$ be a Peano space and let $X$ be a dense, locally connected, connected subset of $P$. Then there exists a $G_\delta$-subset $Y$ of $P$ containing $X$ such that $X$ is locally connected in $Y$ (as an extension of $X$).

**Proof.** Let $n$ be a positive integer and define $Z_n = \{y \in P: \text{if } U \text{ is an open connected subset of } P \text{ containing } y \text{ and } \delta(U) < 2^{-n}, \text{then } U \cap X \text{ is not connected}\}$. (Here $\delta(U)$ denotes the $d$-diameter of $U$). We first assert that $Z_n$ is closed. For suppose $y_1, y_2, \ldots$, is a sequence in $Z_n$ which converges to $y \in (P \setminus Z_n)$. Since $y \notin Z_n$, there exists an open connected subset $U$ of $P$ containing $y$ and $\delta(U) < 2^{-n}$ and $U \cap Z_n \neq \emptyset$ and this is a contradiction. Hence $Z_n$ is closed.

We next assert $Z_n \cap X = \emptyset$. For let $x \in X$ and let $V$ be an open connected subset of $X$ such that $\delta(c_1V) < 2^{-n}$. Then $U = \text{int } c_1V$ is open in $P$ and contains $x$ and $\delta(U) < 2^{-n}$. Furthermore, $U \cap X$ is connected since $V \subseteq U \cap X \subseteq c_1V$ and $V$ is connected. Thus $x \notin Z_n$ and $Z_n \cap X = \emptyset$.

Clearly $Z_1 \subset Z_2 \subset Z_3 \ldots$ is a monotonically increasing sequence and if for each $i \geq 1$, $V_i = P \setminus Z_i$, $Y = \bigcap_{i=1}^{\infty} V_i$ is a connected $G_\delta$-subset of $P$ which contains $X$.

We now assert that $X$ is locally connected in $Y$, as an extension of $X$. For let $\varepsilon > 0$ and let $y \in Y$. Then there exists a positive integer $n$ so that $\varepsilon > 2^{-n}$, and...
and since \( Y \subseteq \mathbb{Z} \), there exists an open connected subset \( U \) of \( P \) with \( \delta(U) < 2^{-n} \) and such that \( U \cap X \) is connected. This implies that \( W = \text{int}_Y \text{cl}_Y U \) is an open connected subset of \( Y \). Thus \( Y \) has a basis consisting of regions whose intersection with \( X \) is connected. This completes the proof.

COROLLARY 3.1.1. Every dense, locally connected, connected \( G_\delta \)-subset of a Peano continuum is \( S \)-metrizable if and only if dense, locally connected, connected subset of a Peano continuum is \( S \)-metrizable.

PROOF. This follows from (2.1), (2.4) and (3.1).

Since every nested intersection of countably many sets can be represented as an inverse limit space and since every \( Y_i \) above is \( S \)-metrizable, by (2.5), we ask:

QUESTION 2. If \( \{Y_i, f_{i,j}, N\} \) is an inverse limit sequence of \( S \)-metrizable spaces and continuous maps (bicontinuous injections), must \( Y_\infty = \text{inv lim} \{Y_i, f_{i,j}, N\} \) be \( S \)-metrizable?

Of course an affirmative answer to Question 2 would yield an affirmative answer to Question 1.

THEOREM 3.2. Let \((X,d)\) be a locally connected, connected separable metric space, let \( \beta X \) denote the Stone-Čech compactification of \( X \). Then \( X \) is \( S \)-metrizable if and only if there exists a Peano compactification \( P \) of \( X \) such that \( \beta f \), the continuous extension of the identity injection \( f:X \to P \) to \( \beta X \), is monotone.

PROOF. Recall that a map between compact Hausdorff spaces is monotone if every point inverse is connected. Suppose that \((X,d)\) is \( S \)-metrizable, say \( \rho \) is an \( S \)-metric for \( X \). By (2.3), there exists a Peano compactification \( P \) of \( X \) and \( X \) is locally connected in \( P \). Let \( \beta f: \beta X \to P \) be the continuous extension of the identity map \( f:X \to P \) to \( \beta X \). We need to show that for \( y \in P \), \( \beta f^{-1}(y) \) is connected. But since \( P \) is a metric space and \( X \) is locally connected in \( P \), there exists a neighborhood basis for \( y \) in \( P \), \( \{U_i\}_{i=1}^\infty \) such that for \( i \in \mathbb{N} \), \( \text{cl} U_{i+1} \subseteq U_i \) and...
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$U_i \cap X$ is connected. Then, if $\beta f^{-1}(U_i) = W_i$, $\beta f^{-1}(U_i \cap X) = f^{-1}(U_i \cap X)$ is connected and $W_i \cap X = \beta f^{-1}(U_i \cap X)$. Thus by (1.4) of [7], $W_i$ is connected. It then follows that $\beta f^{-1}(y) = \bigcap_{i=1}^{\infty} \text{cl} W_i$ is connected and that completes the proof of the necessity.

Now suppose $(P, \rho)$ is a Peano compactification of $X$ and $\beta f: \beta X \to P$ is a monotone map. Let $y \in P$ and let $V$ be an open connected subset of $P$ containing $y$. Since $\beta f$ is monotone, $\beta f^{-1}(V) = W$ is a connected open subset of $\beta X$. Again, by (1.4) of [7], $W \cap X$ is connected. This implies that $\beta f(W \cap X) = f(W \cap X) = V \cap X$ is connected and so $X$ is locally connected in $P$. By (2.3), $S$ is $S$-metrizable.

4. AN EXAMPLE. This is an example which fails to be $S$-metrizable, however it also fails to have a Peano compactification.

Let $L_i$ be the line in $\mathbb{R}^2$ defined by $L_i = \{(x,y): y = x/i, 0 \leq x \leq 1\}$ and let $X = \bigcup_{i=1}^{\infty} L_i$ with the relative topology inherited from $\mathbb{R}^2$. We first assert that $X$ is not $S$-metrizable. For in any (Hausdorff) compactification $Z$ of $X$, $U_i = L_i \setminus \{(0,0)\}$ is an open subset of $Z$ and since $A = \{(0,0)\}$ is compact, $A$ and $B = \bigcup_{i=1}^{\infty} \{(1,1-i^{-1})\}$ are subsets of $X$ whose closures are disjoint in $Z$. Thus if $Z$ is a metric space with metric $r$ and the distance from $A$ to $\text{cl}_Z B$ is $\varepsilon$, then $\varepsilon > 0$. It then follows that no finite collection of connected sets with $r$-diameter less than $\varepsilon/2$ fails to cover $Z$. Thus $r$ is not a Property $S$ metric for $Z$ and $X$ is not $S$-metrizable.

We will now show that $X$ fails to have a locally connected metric compactification. Suppose $(Z, r)$ is a locally connected metric compactification of $X$. Let $U$ and $V$ be open subsets of $Z$ containing $(0,0)$ such that $\text{cl}_U \subseteq V \subseteq (Z \setminus \text{cl} B)$ ($B$ is defined above). Then each $L_i$ intersects $\text{bd} U$ and $\text{bd} V$ and contains a subarc $S_i$ such that $S_i \subseteq (\text{cl} V \setminus U)$ and $S_i$ meets each of $\text{bd} V$ and $\text{bd} U$ in a single point, say $S_i \cap \text{bd} V = \{a_i\}$ and $S_i \cap \text{bd} U = \{b_i\}$. Without loss of generality we may suppose that $(a_i)_{i=1}^{\infty}$ converges to a point $a \in \text{bd} V$ and $(b_i)_{i=1}^{\infty}$ converges to a point $b \in \text{bd} B$. Then $L = \limsup \{S_i: i \in \mathbb{N}\}$ is a connected set subset of $\text{cl} V \setminus U$ meeting $\text{bd} U$ and $\text{bd} V$ [8, p. 14]. Then since every point of $L \setminus (\text{bd} U \cup \text{bd} V)$ is a limit
point of $\bigcup_{i=1}^{\infty} S_i$ and each $S_i$ is a component of $\text{cl} W \setminus U$, $Z$ fails to be locally connected at any point of $L \setminus (\text{bd} U \cup \text{bd} V)$. Thus $X$ fails to have a Peano compactification.

REFERENCES