PROBABILISTIC DERIVATION OF A BILINEAR SUMMATION FORMULA FOR THE MEIXNER-POLLACZEK POLYNOMIALS

P. A. LEE
Department of Mathematics
University of Malaya
Kuala Lumpur 22-11
Malaysia

(Received August 24, 1978)

ABSTRACT. Using the technique of canonical expansion in probability theory, a bilinear summation formula is derived for the special case of the Meixner-Pollaczek polynomials \( \{ \lambda_n^{(k)}(x) \} \) which are defined by the generating function

\[
\sum_{n=0}^{\infty} \lambda_n^{(k)}(x) \frac{z^n}{n!} = (1 + z)^{\frac{1}{2}(x-k)}/(1 - z)^{\frac{1}{2}(x+k)}, \quad |z| < 1.
\]

These polynomials satisfy the orthogonality condition

\[
\int_{-\infty}^{\infty} p_k(x) \lambda_m^{(k)}(ix) \lambda_n^{(k)}(ix) dx = (-1)^n n! (k)_{\frac{5}{2}, m, n}, \quad i = \sqrt{-1}
\]

with respect to the weight function

\[
\begin{align*}
p_1(x) &= \text{sech } \pi x \\
p_k(x) &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \text{sech } \pi x_1 \text{ sech } \pi x_2 \ldots \text{ sech } \pi(x - x_1 - \ldots - x_{k-1}) dx_1 dx_2 \ldots dx_{k-1}, & k = 2, 3, \ldots
\end{align*}
\]
1. INTRODUCTION

Let $U$ be a Cauchy random variable with the probability density function (p.d.f.)

$$f(u) = \frac{1}{\pi} \frac{1}{1 + u^2}, \quad -\infty < u < \infty.$$  

Consider the transformation $U = \sinh \pi V$. The p.d.f. of $V$ is

$$p(v) = \text{sech} \, \pi v, \quad -\infty < v < \infty.$$  \hspace{1cm} (1)

This is the hyperbolic secant distribution considered by Baten [2], and is a special case of the generalized hyperbolic secant distribution treated by Harkness and Harkness [10].

Let $X_1$ and $X_2$ be two random variables having additive random elements in common [6], i.e.

$$X_1 = V_1 + V_2$$

$$X_2 = V_2 + V_3$$

where $V_i$ $(i = 1, 2, 3)$ are mutually independent random variables each having the p.d.f. given in (1). The joint p.d.f. $p(x_1, x_2)$ of $X_1$ and $X_2$ is easily shown to be

$$p(x_1, x_2) = \int_{-\infty}^{\infty} \text{sech} \, \pi z \, \text{sech} \, \pi (x_1 - z) \, \text{sech} \, \pi (x_2 - x_1 + z) \, dz$$

$$= \frac{1}{2} \, \text{sech} \, \frac{\pi x_1}{2} \, \text{sech} \, \frac{\pi x_2}{2} \, \text{sech} \, \frac{\pi (x_2 - x_1)}{2}, \quad -\infty < x_1 < \infty,$$  \hspace{1cm} (2)

$$-\infty < x_2 < \infty.$$

The marginal p.d.f.'s for $X_1$ and $X_2$ are respectively
The orthogonal polynomials with the above marginals as weight function are related to the Euler numbers and have been discussed by Carlitz in [4]. Specifically, for the weight function

\[ p_k(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{sech} \, \pi x_1 \, \text{sech} \, \pi x_2 \, \cdots \, \text{sech} \, \pi (x - x_1 - x_2 - \cdots - x_{k-1}) \, dx_1 \cdots dx_{k-1}, \quad k = 2, 3, \ldots \quad (4) \]

the polynomials \( \left\{ \lambda_n^{(k)}(x) \right\} \) with generating function

\[ \sum_{n=0}^{\infty} \lambda_n^{(k)}(x) z^n / n! = (1 + z)^{\frac{1}{2}(x-k)}/(1 - z)^{\frac{1}{2}(x+k)}, \quad |z| < 1 \quad (5) \]

are the orthogonal polynomials in the interval \((-\infty, \infty)\) and satisfy the orthogonality condition

\[ \int_{-\infty}^{\infty} p_k(x) \lambda_m^{(k)}(ix) \lambda_n^{(k)}(ix) \, dx = (-1)^n n! (k)_n \delta_{m,n} \quad (6) \]

where \( i = \sqrt{-1} \) and \( \delta_{m,n} \) denotes the Kronecker delta.

The explicit form of the orthogonal polynomial is given by Carlitz [4] as

\[ \lambda_n^{(k)}(x) = \sum_{r=0}^{n} 2^r \binom{n+k-1}{n-r} \binom{n+k}{r} \]

\[ = (-1)^n (k)_n \sum_{r=0}^{n} \binom{n+k}{r} \left[ -n, \frac{1}{2}(x+k); 2 \right] \quad (7) \]

The last result follows easily from the following well-known generating function for the Gaussian hypergeometric function \( \sum_{r=0}^{n} \binom{n+k}{r} \left[ -n, \frac{1}{2}(x+k); 2 \right] \).
A related system of polynomials has been discussed by Bateman [1] who referred to them as the Mittag-Leffler polynomials. It happens that both the polynomials discussed by Bateman and Carlitz are but special cases of the system of orthogonal polynomials first discussed by Meixner [11] and later independently by Pollaczek [12]. Following the notation of [8, p. 219] (See also [5, p. 184]), the Meixner-Pollaczek polynomials are given explicitly by

$$p_n^{(\alpha)}(x; \phi) = \frac{(2\alpha n)!}{n!} e^{i\phi} \cdot F_1[-n, \alpha + ix; 2\alpha; 1 - e^{-2i\phi}]$$

(8)

where $\alpha > 0$, $0 < \phi < \pi$ and $-\infty < x < \infty$.

These polynomials are orthogonal with respect to the weight function

$$\omega(\alpha)(x; \phi) = \left(\frac{2 \sin \phi}{\pi}\right)^{2\alpha-1} e^{-(\pi-2\phi)x} |\Gamma(\alpha + ix)|^2.$$

The orthogonality relation is given by

$$\int_{-\infty}^{\infty} \omega(\alpha)(x; \phi) p_m^{(\alpha)}(x; \phi) p_n^{(\alpha)}(x; \phi) dx = \frac{\Gamma(2\alpha + n)}{n!} \csc \phi \delta_{m,n}.$$

A generating function for $p_n^{(\alpha)}(x; \phi)$ is

$$\sum_{n=0}^{\infty} t^n p_n^{(\alpha)}(x; \phi) = (1 - te^{i\phi})^{-\alpha + ix} (1 - te^{-i\phi})^{-\alpha - ix}, \quad |t| < 1.$$

(9)

It is clear when comparing (7) with (8) or (5) with (9) that

$$(-i)^n \lambda_n^{(k)}(ix) = n! p_n^{(k/2)}(x/2; \pi/2), \quad k = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots$$

and thus $\lambda_n^{(k)}(x)$ may be regarded as a special case of the Meixner-Pollaczek polynomials.
2. **A BILINEAR SUMMATION FORMULA**

From the generating function in (5) it is immediately clear that \( \lambda_n^{(k)}(x) \) satisfies the following so-called Runge-type identity

\[
\lambda_n^{(k_1+k_2)}(x_1 + x_2) = \sum_{r=0}^{n} \binom{n}{r} \lambda_r^{(k_1)}(x_1) \lambda_{n-r}^{(k_2)}(x_2), \quad k_1, k_2 = 1, 2, 3, \ldots \quad (10)
\]

and all \( n \).

It has been shown that the result in (10) is both necessary and sufficient for the joint p.d.f. in (2) to possess a bilinear expansion (also called a canonical expansion in statistical literature) of the form [6]

\[
p(x_1, x_2) = g(x_1)h(x_2) \sum_{r=0}^{\infty} \rho_n x_1^n x_2^n
\]

where the canonical variables \( \{\theta_n(x)\} (\{\phi_n(x)\}) \) are a complete set of orthonormal polynomials with weight function \( g(x) \) (h(x)). The canonical correlation is

\[
\rho_n = E[\theta_n(X_1)\phi_n(X_2)]
\]

where \( E \) denotes the expectation operation.

For the joint p.d.f. in (2) with the equal marginal p.d.f.'s given in (3), we note that the canonical variable in this case is

\[
\theta_n(x) = \phi_n(x) = \frac{i^{-n}}{\sqrt{n!(2)^n}} \lambda_n^{(2)}(ix).
\]

The canonical correlation is

\[
\rho_n = E[\theta_n(X_1)\phi_n(X_2)]
\]

\[
= \frac{(-1)^{-n}}{n!(2)^n} E[\lambda_n^{(2)}(i(v_1 + v_2))\lambda_n^{(2)}(i(v_2 + v_3))]
\]

\[
= \frac{(-1)^{-n}}{n!(2)^n} \sum_{s=0}^{n} \sum_{r=0}^{n} \binom{n}{s} \binom{n}{r} E[\lambda_r^{(1)}(iv_1)]
\]


We therefore have the following interesting bilinear summation formula for the Meixner-Pollaczek polynomials

\[
E[\lambda_n^{(1)}(i\nu_2)\lambda_s^{(1)}(i\nu_3)]
= (-1)^n \frac{1}{n!(2)_n} E[\lambda_n^{(1)}(i\nu_2)]^2
= \frac{1}{n+1}.
\]

We therefore have the following interesting bilinear summation formula for the Meixner-Pollaczek polynomials

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{[(n+1)!]^2} \lambda_n^{(2)}(ix_1)\lambda_n^{(2)}(ix_2) = \sinh \frac{\pi x_1}{2} \sinh \frac{\pi x_2}{2} \text{sech} \left(\frac{\pi x_2 - x_1}{2}\right) / (2x_1x_2)
\]

\[\rightarrow < x_1 < \infty \text{ and } \rightarrow < x_2 < \infty.\]  

3. A GENERALIZATION

Consider the following more general scheme of additive random variables as in [9].

Let \(\{\xi_i\}\) for \(i = 1, 2, \ldots, n - m\), \(\{\eta_i\}\) for \(i = 1, 2, \ldots, m\) and \(\{\tau_i\}\) for \(i = 1, 2, \ldots, n_2 - m\) where \(1 \leq m < \min(n_1, n_2)\) be \((n_1 + n_2 - m)\) mutually independent random variables each having the p.d.f. given in (1).

Define

\[
U = \sum_{i=1}^{n_1-m} \xi_i, \quad V = \sum_{i=1}^{m} \eta_i, \quad W = \sum_{i=1}^{n_2-m} \tau_i
\]

\[X_1 = U + V\]

\[X_2 = V + W.\]

It is clear that the joint characteristic function \(\phi(\omega_1, \omega_2)\) of \(X_1\) and \(X_2\) is

\[
\phi(\omega_1, \omega_2) = E[\exp(i\omega_1 X_1 + i\omega_2 X_2)]
\]
\[
E\{\exp[i\omega_1 U + i\omega_2 W + i(\omega_1 + \omega_2)V]\} \\
= \text{sech}^{n_1-m}\left(\frac{\omega_1}{2}\right) \text{sech}^{n_2-m}\left(\frac{\omega_2}{2}\right) \text{sech}\left(\frac{\omega_1 + \omega_2}{2}\right)
\]

since
\[
E\left[e^{i\omega_1 t}\right] = E\left[e^{i\omega_2 t}\right] = E\left[e^{i\omega_k t}\right]
\]

= \int_{-\infty}^{\infty} e^{iv} \text{sech}\left(\frac{v}{2}\right) dv

= \text{sech}\left(\frac{v}{2}\right) \text{ for } 1 \leq i \leq n_1 - m,

1 \leq j \leq m,

1 \leq k \leq n_2 - m.

The joint p.d.f. in question is therefore
\[
p(x_1, x_2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega_1 x_1 - i\omega_2 x_2} \text{sech}^{n_1-m}\left(\frac{\omega_1}{2}\right) \text{sech}^{n_2-m}\left(\frac{\omega_2}{2}\right) \text{sech}\left(\frac{\omega_1 + \omega_2}{2}\right) \text{d}\omega_1 \text{d}\omega_2
\]

and the marginal p.d.f.'s for \(X_1\) and \(X_2\) are respectively
\[
g(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_1 x_1} \text{sech}\left(\frac{\omega_1}{2}\right) d\omega_1
\]

= \frac{1}{\pi \left(\frac{n_1}{2} + ix_1\right)!} \left|\Gamma\left(\frac{n_1}{2} + ix_1\right)\right|^2

\[
h(x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_2 x_2} \text{sech}\left(\frac{\omega_2}{2}\right) d\omega_2
\]

= \frac{1}{\pi \left(\frac{n_2}{2} + ix_2\right)!} \left|\Gamma\left(\frac{n_2}{2} + ix_2\right)\right|^2

on using the fact that [3, p. 31]
\[
\int_{-\infty}^{\infty} e^{-ivx} \left[ \text{sech}(\beta x + \gamma) \right]^{\mu+1} dx = \frac{2^\mu}{\beta} \left[ \frac{\Gamma\left(\frac{1 + \mu + iv/\beta}{2}\right)}{\Gamma(\mu + 1)} \right]^2 e^{iv/\beta}. \quad (14)
\]

The respective canonical variables are

\[
\theta_n(x_1) = \frac{i^{-n}}{\sqrt{n!(n_1)!}} \lambda_n (ix_1)
\]

\[
\phi_n(x_2) = \frac{i^{-n}}{\sqrt{n!(n_2)!}} \lambda_n (ix_2).
\]

By a repeated application of the Runge-type identity in (10) analogous to the derivation leading to the result in (11), it may be shown that the canonical correlation in this case is

\[
\rho_n = \frac{\binom{m}{n}}{\sqrt{n!(n_1)!n_2)!}}, \quad 1 \leq m < \min(n_1, n_2).
\]

On the other hand, note that from (14)

\[
\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_1 x_1 - i\omega_2 x_2} \text{sech} \left( \frac{\omega_1 + \omega_2}{2} \right) d\omega_1 d\omega_2
\]

\[
= \frac{2^{m-1}}{\pi^2} \left[ \Gamma\left(\frac{m}{2} + iv_1\right) \right]^2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-i\omega_2(x_2 - x_1)] d\omega_2
\]

\[
= \frac{2^{m-1}}{\pi^2} \left[ \Gamma\left(\frac{m}{2} + iv_1\right) \right]^2 \delta(x_2 - x_1)
\]

where \(\delta(x)\) denotes the Dirac delta function.

A double convolution operation applied to (13) then yields the following expression for \(p(x_1, x_2)\)

\[
p(x_1, x_2) = \frac{n_1^{n_2+m-3}}{\pi^3(m-1)!n_1!-n_1-m-1)!n_2!-n_2-m-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{n_1 - m}{2} + iu\right) \right|^2 \left| \Gamma\left(\frac{n_2 - m}{2} + iv\right) \right|^2 \delta(x_2 - v - x_1 + u) dudv
\]

\[
\left| \Gamma\left(\frac{n_2 - m}{2} + iv\right) \right|^2 \cdot \left| \Gamma\left(\frac{n_1 - m}{2} + iu\right) \right|^2 \cdot \delta(x_2 - v - x_1 + u) dudv
\]
Finally, the result in (15) may be rewritten into the following Barnes type contour integral

\[
p(x_1, x_2) = \frac{n_1 + n_2 + m - 2}{\pi^2 (m - 1)! (n_1 - m - 1)! (n_2 - m - 1)!} \cdot \frac{1}{2 \pi i} \int_C \frac{\Gamma \left( \frac{n_1 - m}{2} + s \right) \Gamma \left( \frac{m}{2} - i x_1 + s \right) \Gamma \left( \frac{n_2 - m}{2} + i (x_2 - x_1) + s \right)}{\Gamma \left( \frac{n_1 - m}{2} - s \right) \Gamma \left( \frac{m}{2} + i x_1 - s \right) \Gamma \left( \frac{n_2 - m}{2} - i (x_2 - x_1) - s \right)} ds
\]

which may be evaluated in terms of a sum of \( _3F_2 \) functions \( [13, \text{p. 133}] \) or, perhaps more conveniently, in terms of Meijer's G-function as follows \( [7, \text{Sec. 5.3}] \)

\[
p(x_1, x_2) = \frac{n_1 + n_2 + m - 2}{\pi^2 (m - 1)! (n_1 - m - 1)! (n_2 - m - 1)!} \cdot \frac{1}{2 \pi i} \int_C \frac{\Gamma \left( \frac{n_1 - m}{2} + s \right) \Gamma \left( \frac{n_2 - m}{2} + s \right) \Gamma \left( \frac{m}{2} + i x_1 + s \right)}{\Gamma \left( \frac{n_1 - m}{2} - s \right) \Gamma \left( \frac{m}{2} - i x_1 - s \right) \Gamma \left( \frac{n_2 - m}{2} - i (x_2 - x_1) - s \right)} ds
\]

The existence of a diagonal expansion then implies the following summation formula

\[
\sum_{n=0}^{\infty} \frac{(-1)^n (n_1)^n (n_2)^n}{(n_1)_n (n_2)_n} \frac{(x_1)^\lambda_n (x_2)^\lambda_n}{(n_1 - 1)! (n_2 - 1)!} = \frac{(n_1 - m)! (n_2 - m)!}{2^m (m - 1)! (n_1 - m - 1)! (n_2 - m - 1)!} \cdot \frac{1}{\Gamma \left( \frac{n_1 + x_1}{2} \right) \Gamma \left( \frac{n_1}{2} - x_1 \right) \Gamma \left( \frac{n_2 + x_2}{2} \right) \Gamma \left( \frac{n_2}{2} - x_2 \right)}
\]
It is perhaps interesting to note in passing that a comparison of the two results in (12) and (16) allows us to deduce the following special case of the G-function, viz.

\[
\frac{\Gamma^{3,3}_{3,3}}{3,3} \left| \begin{array}{ccc}
1 - \frac{n_1 - m}{2}, & 1 - \frac{m}{2} + x_1, & 1 - \frac{n_2 - m}{2} - (x_2 - x_1) \\
\frac{n_1 - m}{2}, & \frac{m}{2} + x_1, & \frac{n_2 - m}{2} - (x_2 - x_1)
\end{array} \right|
\]

(16)

for \(1 \leq m < \min(n_1, n_2), -\infty < x_1 < \infty, -\infty < x_2 < \infty\).

ACKNOWLEDGEMENT. The author is grateful to the referee for his helpful comments, in particular regarding the connection of the system of orthogonal polynomials discussed by Carlitz with that of the Meixner-Pollaczek polynomials.

REFERENCES


2. Baten, W.D. The Probability Law for the Sum of \(n\) Independent Variables, each subject to the Law \((1/(2\pi)) \text{sech}(\pi x/(2\pi))\), Bull. Amer. Math. Soc. 40 (1934) 284-290.


