ABSTRACT. In this paper we consider a parabolic partial differential system of the form $D_t^\alpha H_t = L(t,x,D) H_t$. The generalized stochastic solutions $H_t$, corresponding to the generalized stochastic initial conditions $H_0$, are given. Some properties concerning these generalized stochastic solutions are also obtained.

KEY WORDS AND PHRASES. Generalized Stochastic Solutions, Strongly Parabolic Systems.

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1. INTRODUCTION.

Consider the system

$$D_t^\alpha u = Lu$$ (1.1)
where

\[ D_t = \frac{\partial}{\partial t}, \quad L = \sum_{|k| \leq 2b} L_k(t,x) D^k, \]

\[ D^k = (-i)^k D_1^{k_1} \cdots D_n^{k_n}, \quad D_r = \frac{\partial}{\partial x_r}, \quad r = 1, \ldots, n, \]

\[ |k| = k_1 + \ldots + k_n, \quad t \in (0,T), \quad T > 0, \quad x \text{ is an element of the } n \text{-dimensional} \]

Euclidean space \( E_n \), and \( (L_k(t,x), |k| \leq 2b) \) is a family of square matrices of order \( N \).

We assume that (1.1) is a strongly parabolic system on \( G_{n+1} = \{(t,x): t \in [0,T], x \in E_n\} \) in the sense that for every complex vector \( a = (a_1, \ldots, a_N) \), every \( \sigma \in E_n \), and every \( (t,x) \in G_{n+1} \);

\[
\Re \left[ \sum_{|k| = 2b} L_k(t,x) \sigma^k a, \overline{a} \right] \leq -\delta |\sigma|^{2b} |a|^2
\]

where

\[ \sigma^k = \sigma_1^{k_1} \cdots \sigma_n^{k_n}, \quad |\sigma|^{2b} = (\sigma_1^{2} + \ldots + \sigma_n^2)^{b}, \]

\[ |a|^2 = a_1^2 + \ldots + a_N^2, \quad \text{and } \delta \text{ is a positive constant (see [1]).} \]

In the above inequality and in the following, we denote the scalar product of two \( N \)-vector functions \( u \) and \( v \) by the bracket notation \( (u,v) \).

As usual, we denote by \( C^m(E_n) \), \( 0 \leq m \leq \infty \), the set of all real-valued functions defined on \( E_n \), which have continuous partial derivatives of order up to and including \( m \) (of order \( < \infty \) if \( m = \infty \)). By \( C^m_o(E_n, N) \) we denote the set of all \( N \)-vector functions \( h = (h_1, \ldots, h_N) \) such that every \( h_r \) is in \( C^m(E_n) \), with compact support, \( r = 1, \ldots, N \). We assume that the elements of the matrices \( L_k(t,x), |k| \leq 2b \), satisfy the following conditions:

(a) They are bounded on \( G_{n+1} \) and satisfy a H"older condition of order \( \alpha \) with respect to \( x \), \( 0 < \alpha \leq 1 \).

(b) For every \( x \in E_n \), they are continuous functions in \( t \in [0,T] \).
(c) For every $t$ in $[0, T]$, they are $C^\infty(E_n)$ functions. Let $u = (u_1, \ldots, u_N)$ satisfy the initial condition
\begin{equation}
  u(x, 0) = u_0(x),
\end{equation}
where $u_0 = (u_{01}, \ldots, u_{0N})$, $[u_{or} \in C(E_n)]$ are bounded on $E_n$, $r = 1, \ldots, N$.

We say that $u$ is of the class $S(E_n)$ if for each $t \in (0, T)$, $D_t u \in C(E_n)$ and $u_r \in C^2b(E_n)$, $r = 1, \ldots, N$.

It has been proved [2] that, under conditions (a) and (b), there exists a fundamental matrix $Z(t, 0, x, y)$ of the system (1.1) such that
\begin{equation}
  u(t, x) = \int_{E_n} Z(t, 0, x, y) u_0(y) \, dy, \quad dy = dy_1 \ldots dy_r
\end{equation}
represents the unique solution of the Cauchy problem (1.1), (1.2) in the class $S(E_n)$.

Let $(V_r : r = 1, \ldots, N)$ be a family of Gaussian random measures in the sense of Gelfand and Vilenkin [2]. Let $g_r$ be a complex-valued function defined on $E_1$. We say that $g_r$ is of the class $K_r$ if the integral
\begin{equation}
  \int_{E_1} |g_r(s)|^2 \, dF_r(s) \text{ exists, where } F_r \text{ is a positive measure such that}
\end{equation}
\begin{equation}
  E[V_r(B_1) V_r(B_2)] = F_r(B_1 \cap B_2)
\end{equation}
for any two Borel sets $B_1$ and $B_2$ on the real line $[r = 1, \ldots, N]$ and $E(\cdot)$ denotes the expectation of $(\cdot)$.

Let $H$ be an $N$-vector of generalized stochastic processes, which associates with every $h \in C^\infty(E_n, N)$ an $N$-vector of random variables defined by
\begin{equation}
  H(h) = (H_1(h), \ldots, H_N(h)), \quad H_r(h) = \int_{E_1} g_{r0}(s) \, dV_r(s),
\end{equation}
where
\[ g_{r_0}(s) = \int_{E_{n+1}} (I_r(x,s), h(x)) \, dx, \]

where \((I_r; r = 1, \ldots, N)\) is a family of \(N\)-vectors of continuous functions on \(E_{n+1}\).

It is assumed also that all the components of \(I_r\) are bounded on \(E_n\), independently of \(s\). Clearly, \(g_{r_{00}}\) is of the class \(K_r\).

The theoretical development in section 2 exhibits the use of formula (1.3) in order to integrate (1.1) when the initial condition is an \(N\)-vector of generalized stochastic processes, which is defined by (1.4). Also, some essential properties are derived in section 3.

2. GENERALIZED STOCHASTIC SOLUTIONS.

An \(N\)-vector \(w(t,x,s)\) of functions is said to be of the class \(C(E_{n+1}, N)\) if, for each \(t\) in \((0,T)\), the components of \(w(t,x,s)\) represent continuous functions of \((x,s)\) on \(E_{n+1}\) and they are bounded on \(E_n\), independently of \(s\). We say that the generalized stochastic vector \(H_t\) is of the class \(V\) if there exists a family \(\{S_r(t,x,s) \in C(E_{n+1}, N), r = 1, \ldots, N\}\) such that, for each \(h\) in \(C_0^\infty(E_n, N), H_t(h)\) can be represented in the form

\[ H_t(h) = \int_{E_1} g(t,s) \, dV(s), \]

\[ g = (g_1, \ldots, g_N), \quad g_r(t,s) = \int_{E_n} (S_r(t,x,s), h(x)) \, dx, \]

\[ H_t(h) = (H_{1t}(h), \ldots, H_{Nt}(h)), \quad H_{rt}(h) = \int_{E_1} g_r(t,s) \, dV(s). \]

It is clear that, for each \(t\) in \((0,T), g_r \in K_r\). The expectation of \(|H_{rt}|^2\) is given by
If \( D_t g_r(t,s) \) exists and belongs to \( K_r \) for each \( t \) in \((0,T)\), then we define

\[
\frac{d}{dt} H_{rt}(h) \text{ by}
\]

\[
\frac{d}{dt} H_{rt}(h) = \lim_{t \to 0} \frac{\Delta g_r(t,s)}{\Delta t} d V_r(s) = \int E_1 \frac{\Delta g_r(t,s)}{\Delta t} d V_r(s),
\]

where \( \Delta g_r(t,s) = g_r(t + \Delta t,s) - g_r(t,s) \) and \( \text{l.i.m.} \) denotes limit in the mean, i.e.

\[
\lim_{t \to 0} \int E_1 \left( \frac{\Delta g_r(t,s)}{\Delta t} - D_t g_r(t,s) \right)^2 d F_r(s) = 0.
\]

Let \( L^* = \sum_{|k| \leq 2b} (-1)^{|k|} D^k L_k^* \), where \((L_k^*, |k| \leq 2b)\) is the family of adjoint matrices to \((L_k, |k| \leq 2b)\). Since the coefficients of the operator \( L \) are \( C^\infty(\mathbb{R}^n) \) functions, it follows that, for every \( h \in C^\infty_0(\mathbb{R}^n,N) \), \( L^* h = h_t \) is also in \( C^\infty_0(\mathbb{R}^n,N) \). We call \( H_t \) a generalized stochastic solution of the system \((1.1)\) if \( H_t \) and \( \frac{dH_t}{dt} \) are of the class \( V \) and

\[
\frac{dH_t}{dt}(h) = H_t^*(h_t)
\]

for every \( h \in C^\infty_0(\mathbb{R}^n,N) \) and \( t \) in \((0,T)\). We assume that

\[
H_0(h) = H(h)
\]

where \( H \) is defined by \((1.4)\).

**Theorem 1:** The Cauchy problem \((2.1), (2.2)\) has a unique generalized stochastic solution \( H_t \) in the class \( V \).

**Proof:** Let \((S_r(t,x,s) : r = 1, \ldots, N)\) be a family of solutions of the system \((1.1)\) with the initial conditions:
Using formula (1.3), one gets

\[ S_r(t, x, s) = \int_{E_n} Z(t, 0, x, y) I_r(y, s) \, dy. \]  

(2.3)

According to the properties of the fundamental matrix \( Z \), we find \( S_r \in C(E_{n+1}, N) \), \( r = 1, \ldots, N \). Set,

\[ H_t(h) = \int_{E_1} g(t, s) \, dV(s) \]

and

\[ g_r(t, s) = \int_{E_n} (S_r(t, x, s), h(x)) \, dx \]

with \( h_1 \in C_0(E_{n+1}, N) \), where \( S_1(t, x, s), \ldots, S_N(t, x, s) \) are defined by (2.3). Since \( S_r \in C(E_{n+1}, N) \), it follows that \( H_t \) is of the class \( V \). Using again the properties of \( Z \), we get

\[ D_t \left( (S_r(t, x, s), h(x)) \right) \, dx = \int_{E_n} (D_t S_r(t, x, s), h(x)) \, dx \]

\[ = \int_{E_n} (S_r(t, x, s), h_t^*(x)) \, dx. \]

The last formula proves that \( D_t g_r \in K_r \).

Now we already have

\[ \frac{d}{dt} H_t(h) = \int_{E_1} \int_{E_n} (S_r(t, x, s), h_t^*(x)) \, dx \, dV(s) = H_t(h_t^*), \]

where \( \frac{d}{dt} H_t \) is of the class \( V \).

We also have

\[ H_0(h) = \int_{E_1} g(0, s) \, dV(s), \]
where
\[
g_r(0,s) = \int_{E_n} (I_r(x,s), h(x)) \, dx.
\]

Thus the existence of the generalized stochastic solution \( H_t \) with the initial condition \( H_0 = H \) is proved. To prove the uniqueness of \( H_t \), it is sufficient to show that the only solution of (2.1) with the initial condition \( H_0(h) = H(h) = 0 \) is \( H_t(h) = 0 \) for every \( h \) in \( C^\infty_o(E_n, N) \) and \( t \) in \( (0,T) \). If \( H_0 = 0 \), then
\[
E |H_{ro}|^2 = \int_{E_1} |g_{ro}(s)|^2 \, dF(s) = 0,
\]
and hence \( g_{ro}(s) = 0 \) on \( E_1 \).

Therefore,
\[
g_{ro}(s) = \int_{E_n} (I_r(x,s), h(x)) \, dx = 0,
\]
which is true for any arbitrary \( h \) in \( C^\infty_o(E_n, N) \), and hence \( I_r(x,s) = 0 \) on \( E_{n+1} \).

Since \( \frac{d}{dt} H_t(h) = H_t(h^*_t) \), it follows that
\[
E \left| \frac{d}{dt} H_{rt}(h) - H_{rt}(h^*_t) \right|^2 = 0;
\]
therefore,
\[
\int_{E_n} (D_{tr}(t,x,s) - L S_r(t,x,s), h(x)) \, dx = 0,
\]
which implies
\[
D_{tr}(t,x,s) = L S_r(t,x,s).
\] (2.4)

We also have
\[
S_r(0,x,s) = 0.
\] (2.5)

The uniqueness of the problem (2.4), (2.5) gives
\[
S_r(t,x,s) = 0,
\] (2.6)
$t \in (0,T), (x,s) \in \mathbb{E}_{n+1}, (r = 1, \ldots, N)$. Using (2.6), one gets $H_t(h) = 0$, for every $h \in C_{0}^{\infty}(\mathbb{E}_n, N)$ and $t \in (0,T)$. This completes the proof.

3. A CONVERGENCE THEOREM.

Let $h_m = (h_{m,1}, \ldots, h_{m,N})$, $m = 1, 2, \ldots$ be a sequence in $C_{0}^{\infty}(G, N)$, where $G$ is a bounded open domain of $\mathbb{E}_n$. Suppose that

$$\lim_{m \to \infty} \int_{G} (h_{m,r}(x) - \omega_r(x))^2 \, dx = 0,$$

where $\omega_r \in L_2(G)$, $r = 1, \ldots, N$ and $L_2(G)$ denotes the set of all Lebesgue measurable square integrable functions on $G$. It is assumed that $\omega_r(x) = 0$ for $x \in G$ where $r = 1, \ldots, N$.

**THEOREM 2:** If $H_t(h_m) = \int g_m(t,s) \, dV(s)$,

then

$$\lim_{m \to \infty} H_t(h_m) = \int \eta(t,s) \, dV(s),$$

where $g_m(t,s) = (g_{m,1}(t,s), \ldots, g_{m,N}(t,s))$,

$$g_{m,r}(t,s) = \int (S_r(t,x,s), h_{m}(x)) \, dx, \eta = (\eta_1, \ldots, \eta_N),$$

$$\eta_{r}(t,s) = \int (S_{r}(t,x,s), \omega(x)) \, dx,$$

and the family $(S_r, r = 1, \ldots, N)$ is defined by (2.3).

**PROOF:** A straightforward application of the Cauchy–Schwarz inequality establishes that

$$\lim_{m \to \infty} \int_{G} (S_r(t,x,s), h_{m}(x)) \, dx = \int_{G} (S_r(t,x,s), \omega(x)) \, dx$$

(3.2)
According to the conditions imposed on the family \((I_r(x,s), r = 1, \ldots, N)\)
and according to the properties of the fundamental matrix \(Z\), we can find a
constant \(A\) such that
\[
|g_m(t,s)| \leq A, \quad (3.3)
\]
for all \(m, s, t \in (0,T)\) and \(r = 1, \ldots, N\). For any positive integers \(\ell\) and \(m\),
we have
\[
E \left| H_{rt}^m(h_m) - H_{rt}(h) \right|^2 = \int \left| g_m^r(t,s) - g_{\ell}^r(t,s) \right|^2 dF_r(s). \quad (3.4)
\]
By a standard argument based on (3.2) and (3.3), the righthand side of (3.4) can
be shown to go to zero. Thus, \(H_{rt}(h_m)\) is a Cauchy sequence. We deduce also
that
\[
\lim_{m \to \infty} \int \left| g_m^r(t,s) - \eta_r(t,s) \right|^2 dF_r(s) = 0.
\]
The last argument leads to the fact that there exists a stochastic process
\(R_r(t)\) such that \(E |R_r(t)|^2 < \infty\) and that
\[
\lim_{m \to \infty} E \left| H_{rt}^m(h_m) - R_r(t) \right|^2 = 0.
\]
Following Doob [3], we find
\[
R_r(t) = \int \eta_r(t,s) dV_r(s),
\]
\[
\eta_r(t,s) = \int (S_r(t,x,s), w(x)) dx.
\]
This completes the proof.

**COROLLARY:** For vector functions \((w = w_1, \ldots, w_N)\) where \(w_r \in L_2(Q)\) and
\(w_r(x) = 0\) for \(x \notin \Omega\), there exists a sequence \((h_m)\) in \(C_0^\infty (E_n, N)\) such that
\[
\lim_{m \to \infty} H_0^m(h_m) = H_0(w),
\]
\[
\lim_{m \to \infty} H_t(h_m) = H_t(w).
\]

The proof can be deduced directly by using theorem 2. (Compare [4]).

REFERENCES


