A NON-LINEAR HYPERBOLIC EQUATION

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ABSTRACT. In this paper the following Cauchy problem, in a Hilbert space $H$, is considered:

$$(I + \lambda A)u'' + A^2 u + [\alpha + M(A^{1/2}u^2)]Au = f$$

$u(0) = u_0$
$u'(0) = u_1$

$M$ and $f$ are given functions, $A$ an operator in $H$, satisfying convenient hypothesis, $\lambda \geq 0$ and $\alpha$ is a real number.

For $u_0$ in the domain of $A$ and $u_1$ in the domain of $A^{1/2}$, if $\lambda > 0$, and $u_1$ in $H$, when $\lambda = 0$, a theorem of existence and uniqueness of weak solution is proved.

KEY WORDS AND PHRASES. Nonlinear Wave Equation, Cauchy Problem, Existence and Uniqueness.

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1. INTRODUCTION.

The physical origin of the problem here considered lies in the theory of vibrations of an extensible beam of length \( \ell \), whose ends are held a fixed distance apart, hinged or clamped, and is either stretched or compressed by an axial force, taking into account the fact that, during vibration, the elements of a beam perform not only a translatory motion, but also rotate; see Timoshenko [9].

A mathematical model for this problem is an initial-boundary value problem for the non-linear hyperbolic equation

\[
\frac{\partial^2 u}{\partial t^2} - \lambda \frac{\partial^4 u}{\partial t^2 \partial \lambda^2} + \frac{\partial^4 u}{\partial \lambda^4} - [\alpha + \int_0^\ell \left( \frac{\partial u}{\partial s} (s,t) \right)^2 ds] \frac{\partial^2 u}{\partial \lambda^2} = 0, \tag{1.1}
\]

where \( u(\lambda,t) \) is the deflection of point \( \lambda \) at time \( t \), \( \alpha \) is a real constant, proportional to the axial force acting on the beam when it is constrained to lie along the \( \lambda \) axis, and \( \lambda \) is a nonnegative constant (\( \lambda^2 = 0 \) means neglecting the rotatory inertia, while \( \lambda > 0 \) means considering it). The non-linearity of the equation is due to considering the extensibility of the beam.

This model, when \( \lambda = 0 \), was treated by Dickey [2], Ball [1] and, in a Hilbert space formulation, by Medeiros [5]. For related problems, see Pohozaev [7], Lions [4], Menzala [6] and Rivera [8].

In this paper, a theorem of existence and uniqueness of weak solution for a Cauchy problem in a Hilbert space \( H \), is proved for the equation

\[
(I + \lambda A)u'' + A^2 u + [\alpha + M(|A^{1/2}u|^2)] Au = f, \tag{1.2}
\]

with suitable conditions on the operator \( A \) and the given functions \( M \) and \( f \).

This paper is divided in three parts. In Part 1, the theorem is stated and existence of a weak solution is proved. In Part 2, its uniqueness is established. Finally, an application is given, in Part 3, when \( H \) is \( L^2(\Omega) \), \( \Omega \) a bounded open set with regular boundary in \( \mathbb{R}^n \), and \( A \) is the Laplace operator \(-\Delta\).
2. **EXISTENCE OF WEAK SOLUTION.**

Let $H$ be a real Hilbert space, with inner product $(\ ,\ )$ and norm $||\ ||$. Let $A$ be a linear operator in $H$, with domain $D(A) = V$ dense in $H$. With the graph norm of $A$, denoted $||\ ||_A$, i.e.

$$||v||^2 = |v|^2 + |Av|^2, \text{ for } v \text{ in } V,$$

$V$ is a real Hilbert space and its injection in $H$ is continuous. We assume this injection compact.

Suppose $A$ self-adjoint and positive, i.e., there is a constant $k > 0$ such that

$$(Av,v) \geq k|v|^2, \text{ for } v \text{ in } V. \quad (2.1)$$

Let $V'$ be the dual of $V$ and $\langle,\rangle$ denote the pairing between $V'$ and $V$. Identifying $H$ and $H'$, it follows that $V \subset H \subset V'$. Injections being continuous and dense, it is known that, for $h$ in $H$ and $v$ in $V$, $\langle h,v \rangle = (h,v)$.

Define $A^2 : V \to V'$ by

$$\langle A^2u, v \rangle = (Au,Av), \text{ for } u, v \text{ in } V. \quad (2.2)$$

It follows that $A^2$ is a bounded linear operator from $V$ into $V'$.

Let $a(u,v)$ denote the bilinear form in $D(A^{1/2})$ associated to $A$, i.e.,

$$a(u,v) = (A^{1/2}u, A^{1/2}v), \text{ for } u, v \text{ in } D(A^{1/2})$$

$a(u)$ means $a(u,u)$.

Given $\lambda \geq 0$, consider in $W = D((\lambda A)^{1/2})$ the graph norm of $(\lambda A)^{1/2}$, denoted $||\ ||_\lambda$, i.e.,

$$||w||^2_\lambda = |w|^2 + \lambda|A^{1/2}w|^2, \text{ for } w \text{ in } W$$

Note that $W = H$, if $\lambda = 0$, and $W = D(A^{1/2})$, if $\lambda > 0$; hence $V$ is dense in $W$.

Let $\alpha$ be a real number, $M$ a real $C^1$ function, with $M'(s) \geq 0$, for $s \geq 0$.

Assume the existence of positive constants $m_o$ and $m_1$ such that $M(s) \geq m_o + m_1s$, 

...
for $s \geq 0$. Notice that, should $M$ be the identity function, replacement of $\alpha + s$ by $(\alpha - \alpha_0) + (\alpha_0 + s)$, with arbitrary $\alpha_0 > 0$, ensures the fulfilment of the above condition on $M$.

The theorem can now be stated.

**THEOREM.** Given $f$ in $L^2(0,T;H)$, $u_0$ in $V$, $u_1$ in $W$, there is a unique function $u = u(t)$, $0 \leq t < T$, such that:

a) $u \in L^\infty(0,T;V)$

b) $u' \in L^\infty(0,T;W)$

c) $u$ is a weak solution of

\[
(I + kA)u'' + A^2u + [\alpha + M(Au)^{1/2}] Au = f,
\]

i.e., for every $v$ in $V$, $u$ satisfies in $\mathcal{D}'(0,T)$:

\[
\frac{d}{dt} [(u'(t),v) + \lambda a(u'(t),v)] + (Au(t),Av) + \\
+ [\alpha + M(a(u(t)))] a(u(t),v) = (f(t),v),
\]

d) The following initial conditions hold:

\[
u(0) = u_0, \quad u'(0) = u_1
\]

Before proving the theorem, some remarks are pertinent.

Equation (2.3a) makes sense, because (a) and (b) above imply that $u$, $Au$, $Au'$, $(I+kA)u'$ belong to $L^\infty(0,T;H)$.

Initial condition (2.4a) makes sense, because it is known (see Lions [3]) that if $u$ and $u'$ are in $L^\infty(0,T;H)$, then

\[
u \text{ belongs to } C^0(0,T;H),
\]

Now, initial condition (2.4b) must be understood.

Remember $u' \in L^\infty(0,T;W)$ implies that $(I+\lambda A)u' \in L^\infty(0,T;V')$, because

\[
<(I + \lambda A)u',v> = (u',v) + \lambda a(u',v), \text{ for } v \text{ in } V
\]
From (2.3a), it follows that \((I + \lambda A)u'' \in L^2(0,T;V')\). The fact that both \((I + \lambda A)u'\) and \((I + \lambda A)u''\) belong to \(L^2(0,T;V')\) ensures that
\[
(I + \lambda A)u' \in C^0(0,T;V')
\] (2.6)

Therefore \((I + \lambda A)u'(0)\) is defined. Given \(u_1\) in \(W\), set \((I + \lambda A)u'(0) = (I + \lambda A)u_1\), in \(V'\). It follows that \(u'(0) = u_1\), because, it will be proved below,
\[
(I + \lambda A)w = 0, \text{ for } w \text{ in } W, \implies w = 0. \tag{2.7}
\]

Indeed, \(V\) being dense in \(W\), there is a sequence \((v_j)_{j \in \mathbb{N}}\) in \(V\) that converges to \(w\) in \(W\), i.e., as \(j \to \infty\),
\[
|w - v_j|_\lambda^2 = |w - v_j|^2 + \lambda a(w - v_j) \to 0
\]
and
\[
0 = <(I + \lambda A)w, v_j> = (w, v_j) + \lambda a(w, v_j)
\]
tends to
\[
(w, w) + \lambda a(w) = |w|_\lambda^2
\]
Hence \(w = 0\).

Proof of Existence:

It will follow Galerkin method. Suppose, for simplicity, \(v\) separable.

Let, then, \((w_j)_{j \in \mathbb{N}}\) be a sequence in \(V\) such that, for each \(m\), the set
\[w_1, \ldots, w_m\] is linearly independent and the finite linear combinations of \(w_1, w_2, \ldots\) are dense in \(V\). Let \(V_m\) denote the finite subspace of \(V\), spanned by \(w_1, \ldots, w_m\).

(i) Approximate Solutions

Search for \(u_m(t) = \sum_{j=1}^{m} g_j(t)w_j\) in \(V_m\), such that, for all \(v\) in \(V_m\),
\[
((I + \lambda A)u_m''(t), v) + (Au_m(t), Av) + [\alpha + M(a(u_m(t)))](Au_m(t), v) = (f(t), v) \tag{2.8}
\]
\[
u_m(0) = u_{1m}, \quad u_m'(0) = u_{1m}' \tag{2.9}
\]
where \( u_{om} \) converges to \( u_0 \) in \( V \) and \( u_{1m} \) to \( u_1 \) in \( W \).

This system of ordinary differential equations with initial conditions has a solution \( u_m(t) \), defined for \( 0 \leq t < t_m \leq T \). It is convenient to emphasize that the matrix \( ((I+\lambda A)w_j, w_i) \), \( i,j=1,\ldots,m \), is invertible, for otherwise the homogeneous system of linear equations

\[
\sum_{j=1}^{m} ((I+\lambda A)w_j, w_i)x_j = 0, \quad i = 1,\ldots,m,
\]

would have a non-trivial solution \( a_1,\ldots,a_m \), hence

\[
\left| \sum_{j=1}^{m} a_j w_j \right|^2 = ((I+\lambda A) \sum_{j=1}^{m} a_j w_j, \sum_{i=1}^{m} a_i w_i) = 0,
\]

a contradiction to the linear independence of \( w_1,\ldots,w_m \).

(ii) **A Priori Estimates**

For \( v = 2u_m'(t) \), (2.8) becomes:

\[
\frac{d}{dt} |u_m'(t)|^2 + \lambda \frac{d}{dt} a(u_m'(t)) + \frac{d}{dt} |Au_m(t)|^2 + \alpha \frac{d}{dt} a(u_m(t)) + \nonumber
\]

\[
+ M(a(u_m(t))) \frac{d}{dt} a(u_m(t)) = 2(f(t),u_m'(t)) = (2.10)
\]

Set \( \mathcal{M}(\sigma) = \int_{0}^{\sigma} M(s)ds \).

We integrate (2.10) from 0 to \( t < t_m \) and obtain:

\[
|u_m'(t)|^2 + \lambda a(u_m'(t)) + |Au_m(t)|^2 + \mathcal{M}(a(u_m(t))) \leq K_m + |\alpha|a(u_m(t)) + \int_{0}^{t} |u_m'(s)|^2 ds,
\]

(2.11)

where \( K_m = |u_{1m}|^2 + \lambda a(u_{1m}) + |Au_{om}|^2 + M(a(u_{om})) + \int_{0}^{T} |f(s)|^2 ds \).

By choice, \( u_{om} \) and \( u_{1m} \) converge respectively to \( u_0 \) in \( V \) and to \( u_1 \) in \( W \) (remember that \( |u_{1m}|^2 = |u_{1m}|^2 + \lambda a(u_{1m}). \))
Therefore, there is a constant $C_0 > 0$, independent of $m$ and greater than $K_m$ such that (2.11) still holds, with $K_m$ replaced by $C_0$.

Now, $M(s) \geq m_0 + m_1 s$ implies $\bar{M}(\sigma) \geq m_0 \sigma + \frac{m_1}{2} \sigma^2$ (2.12)

For $\sigma = a(u_m(t))$, from (2.11), (2.12) and $|\sigma| \leq \frac{|\alpha|}{2m_1} + \frac{m_1}{2} \sigma^2$

one obtains:

$$|u_m'(t)|^2 + \lambda a(u_m'(t)) + |A u_m(t)|^2 + m_0 a(u_m(t)) \leq C + \int_0^t |u_m'(s)|^2 ds,$$ (2.13)

where $C = C_0 + \frac{|\alpha|}{2m_1}$, a constant independent of $m$.

In particular,

$$|u_m'(t)|^2 \leq C + \int_0^t |u_m'(s)|^2 ds.$$

Hence, applying Gronwall inequality

$$|u_m'(t)|^2 \leq C e^T,$$ (2.14)

It follows from (2.13) and (2.14) that

$$|u_m'(t)|^2 + \lambda a(u_m'(t)) + |A u_m(t)|^2 + m_0 a(u_m(t)) \leq K,$$ (2.15)

where $K = C(1 + T e^T)$, for all $t$ in $[0, t_m]$ and all $m$.

In particular, as $k |u_m(t)|^2 \leq a(u_m(t))$, it follows that $u_m(t)$ remains bounded; hence it can be extended to $[0, T]$. Therefore, (2.15) holds, in fact, for all $m$ and $t$ in $[0, T]$.

(iii) Passage to the Limit

It follows that there is a sub-sequence of $(u_m)$, still denoted $(u_m)$, for which, as $m \to \infty$, the following is true, in the weak star convergence of $L^\infty(0, T; H)$:

$$u_m \rightharpoonup u,$$ (2.16)

$$a(u_m) \rightharpoonup a(u),$$ (2.17)
\begin{align}
    & Au_m \rightarrow Au, \\
    & u_m' \rightarrow u', \\
    & \lambda a(u_m') \rightarrow \lambda a(u'), \\
    & M(a(u_m'))Au_m \rightarrow \psi
\end{align}

It must still be proved that, in fact

\begin{equation}
    \psi = M(a(u))Au - \tag{2.22}
\end{equation}

will be shown to follow from the Lemma below, whose proof, here reproduced, was given by J.L. Lions [3] and [4].

**Lemma.** The mapping \( v \rightarrow M(a(v))Av \) from \( V \) into \( H \) is monotonic.

**Proof.** The function \( \bar{H}(\sigma) = \int_0^\sigma M(s)ds \) is non-decreasing (because \( M'(\sigma) = M(\sigma) \geq 0 \)) and convex (because \( \bar{H}''(\sigma) = M'(\sigma) \geq 0 \)).

Take

\[ \phi(v) = \bar{H}(a(v)), \text{ for } v \text{ in } V \]

It is easy to see that \( \phi \) has a Gateau derivative,

\[ \phi'(v) = 2M(a(v))Av, \text{ for } v \text{ in } V, \]

and that \( \phi \) is convex, i.e., for \( 0 \leq \rho \leq 1 \),

\[ \phi(\rho v + (1-\rho)w) \leq \rho \phi(v) + (1-\rho)\phi(w), \text{ for } v, w \text{ in } V. \]

This inequality can be written in the form

\[ \frac{\phi(w + \rho(v-w)) - \phi(w)}{\rho} \leq \phi(v) - \phi(w) \]

Passing to the limit, as \( \rho \to 0 \) it follows that

\[ (\phi'(w), v-w) \leq \phi(v) - \phi(w) \]

and, interchanging the roles of \( v \) and \( w \),

\[ (\phi'(v), w-v) \leq \phi(w) - \phi(v) \]

Adding the two inequalities above, one obtains:

\[ (\phi'(w) - \phi'(v), w-v) \geq 0, \]
This proves the Lemma.

It can now be shown that (2.22) holds.

Indeed, because of the Lemma, for all \( v \) in \( L^2(0,T;V) \), it is true that

\[
\int_0^T (M(a(u_m))A u_m - M(a(v))Av), u_m - v)dt \geq 0
\]

Because \( (u_m) \) is bounded in \( L^\infty(0,T;V) \) and \( (u'_m) \) in \( L^\infty(0,T;H) \) and the injection of \( V \) in \( H \) is compact, \( (u_m) \) can, further, be supposed to converge to \( u \) strongly in \( L^2(0,T;H) \). Hence, as \( m \to \infty \):

\[
\int_0^T (\psi - M(a(v))Av, u - v)dt \geq 0
\]

Set \( u - v = \rho w, \rho \geq 0 \), divide the inequality by \( \rho \) and let \( \rho \to 0 \), to obtain:

\[
\int_0^T (\psi - M(a(u))Au, w)dt \geq 0
\]

This holds for all \( w \) in \( L^\infty(0,T;V) \), hence \( \psi = M(a(u))Au \).

In the following, let \( k \) be fixed, \( k < m \); take \( v \) in \( V_k \) and let \( m \to \infty \).

(2.19) and (2.20) imply that, in \( D'(0,T) \),

\[
\frac{d}{dt} (u'_m(t),v) \to \frac{d}{dt}(u'(t),v), \tag{2.23}
\]

\[
\lambda \frac{d}{dt} a(u'_m(t),v) \to \lambda \frac{d}{dt} a(u'(t),v) \tag{2.24}
\]

Passing to the limit in (2.8), then (2.23) and (2.24), with (2.17), (2.18), (2.21) and (2.22) ensure that

\[
\frac{d}{dt} [(u'(t),v) + \lambda a(u'(t),v)] + (Au(t),Av) +
\]

\[
+ [a + M(a(u(t)))] a(u(t),v) = (f(t),v), \tag{2.25}
\]

holds in \( D'(0,T) \), for all \( v \) in \( V_k \). By density, (2.25) holds in \( D'(0,T) \), for all \( v \) in \( V \).
Therefore, \( u \) is, indeed, a weak solution of (2.3a).

It must still be shown, in order to complete the proof of existence, that \( u \) satisfies (2.4ab).

(iv) Initial Conditions

(2.19) means that, for \( v \) in \( V \) and \( \theta \) in \( C'(0,T) \) such that \( \theta(0) = 1 \) and \( \theta(T) = 0 \), as \( m \to \infty \)

\[
\int_0^T (u'_m(t),v)\theta(t)dt \to \int_0^T (u'(t),v)\theta(t)dt \tag{2.26}
\]

Because of (2.5) and (2.16), integrating (2.26) by parts, it follows that

\[
(u_{om}, v) \to (u(0), v), \text{ for } v \text{ in } V. \tag{2.27}
\]

But \( u_{om} \to u_0 \) in \( V \); hence (2.27) yields

\[
(u_0, v) = (u(0), v), \text{ for } v \text{ in } V, \text{ i.e., } u \text{ satisfies (2.4a)}.
\]

To show that \( u \) satisfies (2.4b), consider equations (2.3b) and (2.8) for \( v = w_j, j = 1,2,... \). It follows, using (2.17)-(2.18), (2.21) and (2.22), that, as \( m \to \infty \)

\[
\frac{d}{dt} \left[ (u'_m(t), w_j) + \lambda a(u'_m(t), w_j) \right] \tag{2.28}
\]

converges to \( \frac{d}{dt} \left[ (u'(t), w_j) + \lambda a(u'(t), w_j) \right] \) weak star in \( L^\infty(0,T) \).

(2.28) means that for \( v \) in \( V \), \( \theta \) in \( C^1(0,T) \) such that \( \theta(0)=1, \theta(T) = 0 \),

\[
\int_0^T \frac{d}{dt} \left[ (u'_m(t), w_j) + \lambda a(u'_m(t), w_j) \right] \theta(t)dt \to \int_0^T \frac{d}{dt} \left[ (u'(t), w_j) + \lambda a(u'(t), w_j) \right] \theta(t)dt. \tag{2.29}
\]

Because of (2.6), (2.19) and (2.20), integrating (2.29) by parts, it follows that
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\[(u_{lm}, w_j) + \lambda a(u_{lm}, w_j) \rightarrow (u'(0), w_j) + \lambda a(u'(0), w_j), \quad (2.30)\]

But \(u_{lm} \rightarrow u_1\) in \(W\), hence the left-hand side of (2.30) converges also to \((u_1, w_j) + \lambda a(u_1, w_j)\). Therefore

\[(u'(0), w_j) + \lambda a(u'(0), w_j) = (u_1, w_j) + \lambda a(u_1, w_j) \quad (2.31)\]

As (2.31) holds for \(j = 1, 2, \ldots\), it follows that, in fact, for all \(v\) in \(V\):

\[(u'(0), v) + \lambda a(u'(0), v) = (u_1, v) + \lambda a(u_1, v).\]

In other words

\[(I + \lambda A)u'(0) = (I + \lambda A)u_1 \quad \text{in } V'.\]

But this implies, (see [6]) \(u'(0) = u_1\); i.e. \(u\) satisfies (2.4b).

3. UNIQUENESS

Let \(u\) and \(\tilde{u}\) be two solutions of (2.3a) with the same initial conditions (2.4ab). Thus \(w = u - \tilde{u}\) satisfies

\[(I + \lambda A)w'' + A^2 w + A w + M(a(u))w + [M(a(u)) - M(a(\tilde{u}))] \tilde{A}u = 0, \quad (3.1)\]

\[w(0) = 0, \quad w'(0) = 0, \quad (3.2ab)\]

The standard energy method cannot be used to prove uniqueness, because, while the left-hand side of (3.1) belongs to \(L^2(0, T; V')\), \(u'\) belongs to \(L^\infty(0, T; W)\) and not to \(L^\infty(0, T; V)\). A modification has to be made; this procedure can be found in Lions [3].

Consider:

\[z(t) = \begin{cases} 
-\int_t^s w(\xi) d\xi & \text{for } t \leq s \\
0 & \text{for } t > s 
\end{cases} \quad (3.3)\]

and

\[w_1(t) = \int_0^t w(\xi) d\xi, \quad (3.4)\]
Then

\[ z(t) = w_1(t) - w_1(s), \quad (3.5) \]

\[ z(0) = -w_1(s), \quad (3.6) \]

\[ z(s) = 0, \quad (3.7) \]

and

\[ z'(t) = w(t). \quad (3.8) \]

As \( w \in L^\infty(0,T;V) \), it is clear [see (3.3) and (3.8)] that \( z \) and \( z' \) are in

\( L^1(0,T;V) \). Hence, it follows from (3.1) that

\[ \int_0^S <(I + \lambda A)w''(t), z(t)> dt + \int_0^S (Aw(t), Az(t)) dt + \]

\[ + \alpha \int_0^S (Aw(t), z(t)) dt + \int_0^S M(a(u(t)))(Aw(t), z(t)) dt + \]

\[ + \int_0^S [M(a(u(t))) - M(a(\tilde{u}(t)))](A\tilde{u}(t), z(t)) dt = 0. \quad (3.9) \]

But, [see (3.8)]

\[ <(I+\lambda A)w''(t), z(t)> = \frac{d}{dt}((I+\lambda A)w'(t), z(t)) - ((I+\lambda A)w'(t), z'(t)) \]

\[ \frac{d}{dt}((I+\lambda A)w'(t), z(t)) - \frac{1}{2} \frac{d}{dt}((I+\lambda A)w(t), w(t)) \]

Therefore, using (3.2ab) and (3.7), it follows that (remember \( |w|_\lambda^2 = |w|_\lambda^2 + \lambda a(w) \))

\[ \int_0^S <(I + \lambda A)w''(t), z(t)> dt = -\frac{1}{2} |w(s)|^2. \quad (3.10) \]

Now, [see (3.8)]

\[ (Aw(t), Az(t)) = (Az'(t), Az(t)) = \frac{1}{2} \frac{d}{dt} |Az(t)|^2 \]

Thus, [see (3.6) and (3.7)]

\[ \int_0^S (Aw(t), Az(t)) dt = -\frac{1}{2} |Aw_1(s)|^2 \quad (3.11) \]
As \( |w|_\lambda \geq |w| \), (3.9), (3.10) and (3.11) yield

\[
|w(s)|^2 + |A\omega_1(s)|^2 \leq 2|\alpha| \int_0^S (w(t), Az(t)) dt
\]

\[
+ 2 \int_0^S M(a(u(t))) |(w(t), Az(t))| dt
\]

\[
+ 2 \int_0^S |M(a(u(t))) - M(a(\bar{u}(t)))| |(\bar{u}(t), Az(t))| dt
\]

(3.12)

As \( u, \bar{u} \in L^\infty(0,T;V) \) and, for \( s \geq 0, M \geq 0 \) is a \( C^1 \) function, with \( M^1 \geq 0 \), there is a constant \( C > 0 \) such that

\[
2 \int_0^S M(a(u(t))) |(w(t), Az(t))| dt \leq 2C \int_0^S |w(t)| |Az(t)| dt
\]

(3.13)

And

\[
2 \int_0^S |M(a(u(t))) - M(a(\bar{u}(t)))| |(\bar{u}(t), Az(t))| dt
\]

\[
\leq 2C \int_0^S |a(u(t)) - a(\bar{u}(t))| |\bar{u}(t)| |Az(t)| dt
\]

\[
\leq 2C^2 \int_0^S |(A(u(t) + \bar{u}(t)), w(t))| |Az(t)| dt
\]

\[
\leq 2C^3 \int_0^S |w(t)| |Az(t)| dt
\]

(3.14)

Notice that, [see (3.5)]

\[
2|w(t)| |Az(t)| \leq 2[|w(t)|^2 + |A\omega_1(t)|^2] + |A\omega_1(s)|^2.
\]

(3.15)

Hence, it follows from (3.15) that

\[
2|\alpha| \int_0^S (w(t), Az(t)) | dt \leq 2|\alpha| \int_0^T [w(t)|^2 + |A\omega_1(t)|^2] dt + |\alpha| |s| |A\omega_1(s)|^2
\]

(3.16)

Hence (3.13) and (3.15) give
Now (3.14) and (3.15) give

\begin{equation}
2 \int_{0}^{s} M(a(u(t))) \left| (w(t), Az(t)) \right| dt \\
\leq 2C_{0} \int_{0}^{s} \left[ |w(t)|^2 + |A_{w_1}(t)|^2 \right] dt + C_{0} s |A_{w_1}(s)|^2
\end{equation}

(3.17)

It now follows from (3.12), with (3.16) - (3.17) that

\begin{equation}
|w(s)|^2 + (1 - Cs)|A_{w_1}(s)|^2 \leq 2C \int_{0}^{s} \left[ |w(t)|^2 + |A_{w_1}(t)|^2 \right] dt,
\end{equation}

(3.19)

where \( C = |a| + C_{0} + C_{0}^3 \).

Take \( s_0 \) such that, for \( 0 \leq s \leq s_0 \), \( \frac{1}{2} \leq 1 - Cs \leq 1 \). Hence (3.19) yields

for \( 0 \leq s \leq s_0 \):

\[ |w(s)|^2 + \frac{1}{2} |A_{w_1}(s)|^2 \leq 2C \int_{0}^{s} \left[ |w(t)|^2 + |A_{w_1}(t)|^2 \right] dt. \]

A fortiori, for \( 0 \leq s \leq s_0 \),

\[ |w(s)|^2 + |A_{w_1}(s)|^2 \leq 4C \int_{0}^{s} \left[ |w(t)|^2 + |A_{w_1}(t)|^2 \right] dt. \]

Applying Gronwall inequality, it then follows that

\[ w(s) = 0, \text{ for } 0 \leq s \leq s_0. \]

Similarly, it is proved that \( w(s) = 0, \text{ for } s_0 \leq s \leq s_0 + \tau, \text{ with } \tau > 0. \) It then follows that, in fact, \( w(s) = 0, \text{ for } 0 \leq s < T. \)

The proof of uniqueness is complete.
4. APPLICATION

For $\Omega$ a bounded open set in $\mathbb{R}^n$, with regular boundary, consider

$$H = L^2(\Omega), \quad V = H^1_0(\Omega) \cap H^2(\Omega)$$

Let $\Delta$ be the Laplace and $V$ the gradient operators in $\mathbb{R}^n$ respectively. Take $A = -\Delta$, hence $A^\frac{1}{2} = V$. Hypothesis on $A$ are satisfied. Notice that, in this case, the condition $(Av,v) \geq k|v|^2$, for $v$ in $V$, is the Friedrichs - Poincaré inequality; the compactness of the injection of $V$ in $H$ is the Rellich theorem.

It is clear that

$$W = L^2(\Omega), \text{ if } \lambda = 0$$
$$W = H^1(\Omega), \text{ if } \lambda > 0$$

Now $(,) and | \cdot |$ are respectively the inner product and the norm in $L^2(\Omega)$.

Given

$$u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$$
$$u_1 \in L^2(\Omega), \text{ if } \lambda = 0; \quad u_1 \in H^1(\Omega), \text{ if } \lambda > 0,$$

$$f \in L^2(0,T; L^2(\Omega)),$$

the theorem proved above ensures existence and uniqueness of weak solution for the non-linear hyperbolic equation

$$(I - \lambda \Delta)u + \Delta^2 u - [\alpha M(|Vu|^2)] \Delta u = f,$$

satisfying $u(0) = u_0$, $u'(0) = u_1$.

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